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On the time dependence of the wave resistance of a body accelerating from rest

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We consider the large-time behaviour of the disturbances associated with an initial acceleration of a body in or near a free surface. As a canonical problem, we study the case of a body starting impulsively from rest to a constant velocity U. For a constant-strength translating point source (or dipole) started impulsively, it is known that the unsteady part of the Green function oscillates at the critical frequency $\omega_c = g/4U$ (g is the gravitational acceleration) with an amplitude that decays with time, t, as $t^{-1/2}$ and t^{-1} for $t \gg 1$ in two (Havelock 1949) and three (Wehausen 1964) dimensions respectively. These classical results turn out to be non-realistic in that for an actual body, the associated source strengths are in general time dependent and *a priori* unknown, and must together satisfy the kinematic condition on the body boundary. We consider such an initial-boundary-value problem using a transient wave-source distribution on the body surface. Through asymptotic analyses, the unsteady behaviour of the solution at large time is obtained explicitly. Specifically, we show that for a general class of bodies satisfying a simple geometric condition ($\Gamma \neq 0$), the decay rate of the transient oscillations (at frequency ω_c) in the wave resistance and velocity potential is an order of magnitude faster: as $t^{-3/2}$ and t^{-2}, as $t \to \infty$, in two and three dimensions respectively. For body geometries satisfying $\Gamma = 0$, for which the single source is a special case, the classical Green function results are recovered. These results are confirmed by an analysis in the frequency domain and substantiated by direct time-domain numerical simulations.

1. Introduction

The question of how rapidly transients associated with the abrupt motions of a floating body decay in time is one of fundamental theoretical interest as well as practical importance. The rates at which oscillations vanish and measurements are taken are of major concern in model tests especially for unsteady and local effects. The question of the behaviour of transients also arises in almost all numerical simulations in the time domain since it directly affects our ability to extract steady-state predictions for wave-resistance problems or to obtain meaningful results for general seakeeping computations (e.g. Beck & Magee 1990; Lin & Yue 1990; Bingham 1994).

The canonical problem is that of a body in or near a free surface accelerating abruptly from rest to a constant speed U. The main interest is the asymptotic behaviour of the transient solution (such as wave resistance or wave elevation) for large time. Despite its importance, the problem appears to have been addressed only for the idealized case of a single translating source of known strength. Havelock (1949) considered the two-dimensional problem of the wave resistance of a submerged circular cylinder started impulsively from rest. By approximating the body as a point dipole of constant strength, he derived a closed-form solution for the wave resistance. The key finding is that for a given forward speed U, the resistance oscillates about the steady value with frequency $\omega_c = g/4U$, where g is the gravitational acceleration, and the oscillation decays at a rate of $t^{-1/2}$ as $t \to \infty$. This result was extended to three dimensions by Wehausen (1964) who considered a constant source and obtained that the unsteady oscillatory part of the resistance vanishes like t^{-1} for $t \gg 1$.

The question remains as to whether these results are valid for the transient resistance of an actual body. Representing the solution by a wave-source distribution on the body surface, it is clear that the source strengths are in general unsteady and 'coupled' in time through the evolving history of the waves. One observes also that the sources must combine to satisfy a simple condition of no flux on the body surface at all times.

It is not surprising that the time-domain problem of the asymptotic decay of the (ω_c) oscillatory solution is in fact related to the frequency-domain seakeeping problem in the neighbourhood of this frequency, the so-called critical frequency corresponding to $\tau = U\omega_c/g = \frac{1}{4}$. This latter problem was considered by Liu & Yue (1993, hereinafter denoted as LY) for the case of a general body. The time-domain analysis for the long-time behaviour is, however, appreciably more complicated due to the convolution of the unknown source strength evolutions, while the problem is effectively uncoupled in the frequency domain for each frequency.

In this work, we start with the initial-boundary-value problem for the velocity potential (§2) and construct its solution in terms of a distribution on the body of transient wave sources (§4). The time-varying translating wave-source Green function is known from classical theory (Wehausen & Laitone 1960). The evolution of the unknown source distribution is then governed by a time-dependent integral equation resulting from satisfaction of the body boundary condition. Taking into account the large-time asymptotic behaviour of the Green function (§3), the time dependencies of the source distribution as well as the velocity potential are obtained from the integral equation. For a general class of bodies satisfying a simple geometric condition ($\Gamma \neq 0$, see (4.21)), the analysis shows that the asymptotic behaviour of the transient solution is $O(t^{-1}, t^{-3/2}e^{-i\omega_c t})$ and $O(t^{-2}, t^{-2}e^{-i\omega_c t})$, as $t \to \infty$, for two- and three-dimensional bodies respectively. If $\Gamma = 0$, the decay rate is an order-of-magnitude slower and is identical to that for a single constant-strength wave source. For simplicity, the analysis in §4 is carried out in detail for a two-dimensional body only. The extension to three dimensions follows in a straightforward manner and is outlined at the end of §4.

These results in §4 can be anticipated from the associated seakeeping problem in the frequency domain. In that problem, the solution for a single source (the oscillatory-strength constant-forward-speed Green function) is singular at the critical frequency corresponding to $\tau = U\omega_c/g = \frac{1}{4}$ (Haskind 1954). This is consistent with the decay rates of the ω_c oscillations of the time-domain Green functions. LY performed an asymptotic analysis in the neighbourhood of $\tau = \frac{1}{4}$ for a body, and they showed that the solution is bounded at $\tau = \frac{1}{4}$ for a general class of geometries satisfying the same condition ($\Gamma \neq 0$). By considering Fourier transforms between the frequency and time domains, the present results are recovered in §5 from the known frequency-domain behaviour.

The frequency domain perhaps offers an easier argument for the fundamental difference between the solution for a body and the single Green function. That a finite solution for a body exists at $\tau = \frac{1}{4}$ can be reasoned as follows. The induced velocity at any point s, due to a source distribution $\sigma(s')$ on the body, contains a free-surface part $V_\sigma(s)$ which depends on the non-Rankine portion of the Green function. For $\delta^2 \equiv |4\tau - 1| \ll 1$, one can show from the asymptotics that $V_\sigma(s) \sim \alpha_\sigma f(s)$, where α_σ is associated with a Kochin function and $f(s)$ is a property of the geometry (and frequency) but is independent of σ. As $\delta \to 0$, $f(s)$ becomes unbounded (everywhere) like δ^{-1}. In order for a body boundary condition (with finite forcing) to be satisfied as $\delta \to 0$, it follows that $\alpha_\sigma \sim O(\delta)$ for finite σ. Based on source-distribution boundary-integral equations on the body, LY derived the solution of α_σ which can be formally expressed as $\alpha_\sigma/\delta = F/\Gamma$ as $\delta \to 0$ (see LY's equations (3.6) and (4.4)), where F depends on the forcing on the body and Γ is a function of the body geometry (and frequency) only. For a finite forcing, $F = O(1)$ in general and thus $\alpha_\sigma/\delta = O(1)$ if $\Gamma \neq 0$. Otherwise, the solution is singular.

Finally, in §6, we offer numerical evidence for two geometries: (*a*) a submerged (two-dimensional) circular cylinder, and (*b*) a (three-dimensional) Wigley hull; we use two different time-domain computational methods. The direct simulation results are well predicted by the present asymptotic analysis.

2. The initial-boundary-value problem

We consider the time-dependent wave resistance of a floating body accelerating impulsively from rest to a constant velocity U. We choose a right-handed Cartesian coordinate system o-xyz with the (x, y)-plane in the undisturbed free surface, the x-axis pointing in the direction of the velocity U, and the z-axis vertically upwards. This coordinate system translates at forward speed U and is fixed in the body for time $t \geqslant 0$.

We assume the fluid to be incompressible, homogeneous and inviscid, and its motion irrotational. The flow can then be described by a velocity potential:

$$\Phi^*(\boldsymbol{x}, t) = -Ux + \Phi(\boldsymbol{x}, t) \tag{2.1}$$

where Φ represents the body disturbance potential. The potential, Φ, satisfies Laplace's equation, $\nabla^2\Phi = 0$, within the fluid and vanishes at deep water, $\nabla\Phi \to 0$ as $z \to -\infty$. For small surface waves, the linearized free-surface condition can be written as:

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 \Phi + g\frac{\partial \Phi}{\partial z} = 0 \qquad \text{on} \quad z = 0, \tag{2.2}$$

where g is the gravitational acceleration. The kinematic boundary condition applied on the submerged body surface, S_B, is

$$\frac{\partial \Phi}{\partial n} = \begin{cases} 0, & t \leqslant 0 \\ Un_x, & t > 0 \end{cases} \qquad \text{on} \quad S_B, \tag{2.3}$$

where $\boldsymbol{n} = (n_x, n_y, n_z)$ is the unit normal out of the body. At $t=0$, the initial conditions are

$$\Phi(\boldsymbol{x}, 0) = \frac{\partial}{\partial t}\Phi(\boldsymbol{x}, 0) = 0 \qquad \text{on} \quad z = 0. \tag{2.4}$$

The initial-boundary-value problem for Φ is completed with the imposition of an appropriate radiation condition: in this case a physical requirement that waves do not appear far upstream of the body.

We note that the steady-state, i.e. frequency-domain, formulation of the above problem, the so-called Kelvin–Neumann problem, is shown to possess a unique solution (Kochin 1937; Dern 1980; Maz'ya & Vainberg 1993; Quenez 1995) for the case of a deeply submerged body. The extension of this result to more general geometries is, however, not yet available. In the present initial-boundary-value problem context, the fundamental issue is the large-time behaviour of the solution, which is the subject of the present analysis. In particular, it is *a priori* unknown whether the solution is bounded as $t \to \infty$ (see De Prima & Wu 1956; Akylas 1984). We shall show (§4) that for the present case of a body impulsively started and reaching a steady velocity, the solution is bounded and the transient oscillations decay at a much more rapid rate than that of a constant-strength single source.

3. Single-source solutions

We construct the solution of the general problem using a wave-source distribution on the body surface. In order to understand the behaviour, particularly at large time, we pursue in this section the solutions of the single source and their asymptotic behaviour for $t \gg 1$.

Consider a single point source of general time-dependent source strength $\sigma(\tau)$, $-\infty < \tau \leqslant t$, fixed in the (moving) frame at $\boldsymbol{x}'$. The associated velocity potential at a field point $\boldsymbol{x}$ is denoted by $\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma(\tau))$. The potential Ψ is harmonic everywhere in the fluid except at the source position. In addition, Ψ vanishes at large depth and satisfies the linearized free-surface condition (2.2) as well as the far-field radiation condition. The solution Ψ can be derived using classical transform techniques (see Wehausen & Laitone 1960).

3.1. *Two-dimensional sources*

In two dimensions, we write the solution of Ψ in the conventional form:

$$\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma(\tau)) = \sigma(t) \ln\left(\frac{r}{r_1}\right)$$
$$-2g \int_0^\infty \frac{e^{k(z+z')}}{(gk)^{1/2}}\, dk \int_{-\infty}^t \sigma(\tau) \cos[k(x-x') + kU(t-\tau)] \sin[(gk)^{1/2}(t-\tau)]\, d\tau, \quad (3.1)$$

where $r^2, r_1^2 = (x-x')^2 + (z \mp z')^2$. For later analyses, we rewrite (3.1) by expressing the trigonometric functions in complex exponential form:

$$\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma) = \sigma(t) \ln\left(\frac{r}{r_1}\right) + i\frac{g}{2} \int_0^\infty \frac{dk}{(gk)^{1/2}} e^{k\psi + i\Omega_1(k)t} \int_{-\infty}^t \sigma(\tau) e^{-i\Omega_1(k)\tau}\, d\tau + \text{c.c.}$$
$$-i\frac{g}{2} \int_0^\infty \frac{dk}{(gk)^{1/2}} e^{k\psi + i\Omega_2(k)t} \int_{-\infty}^t \sigma(\tau) e^{-i\Omega_2(k)\tau}\, d\tau + \text{c.c.}, \quad (3.2)$$

where $\psi = i(x-x') + (z+z')$, $\Omega_{1,2}(k) = kU \pm (gk)^{1/2}$, and c.c. denotes the complex conjugate of the preceding term. Changing variable with $k = m^2$, and manipulating the range of integration, it follows that

$$\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma) = \sigma(t) \ln\left(\frac{r}{r_1}\right) + i2g^{1/2} \int_0^\infty e^{m^2\psi + i\Omega_1(m)t}\, dm \int_{-\infty}^t \sigma(\tau) e^{-i\Omega_1(m)\tau}\, d\tau + \text{c.c.}$$
$$-ig^{1/2} \int_{-\infty}^\infty e^{m^2\psi + i\Omega_2(m)t}\, dm \int_{-\infty}^t \sigma(\tau) e^{-i\Omega_2(m)\tau}\, d\tau + \text{c.c.} \quad (3.3)$$

Since our objective is to obtain the behaviour of solutions near the body (e.g. for the wave resistance or wave elevations) at large time, it is useful to expand in advance the single-source solution Ψ for $|x - x'|/Ut = o(1)$ for $t \gg 1$. Given $\sigma(\tau)$, the asymptotic expansion of Ψ can be obtained from (3.3) using the method of steepest descent. For later use, we summarize the results for three special cases of $\sigma(t)$. The detailed derivations are given in Appendix A.

3.1.1. *Case I: $\sigma(\tau) = 0$ for $\tau \leqslant 0$, $\sigma(\tau) = 1$ for $0 < \tau \leqslant t$*

For a source of constant strength 1 for $0 < \tau \leqslant t$, the potential $\Psi(x, t; x', \sigma) \equiv \Psi_1(x, t; x')$ at large time t can be expanded as

$$\Psi_1 = G_1(x; x') + C_1 \frac{e^{-i\omega_c t}}{t^{1/2}} e^{\kappa\varphi} + \text{c.c.} + O\left(\frac{e^{-i\omega_c t}}{t^{3/2}}\right), \qquad t \to \infty, \tag{3.4}$$

where the wavenumber $\kappa \equiv g/(4U^2)$ and the constant $C_1 = 8(\pi U/g)^{1/2} e^{i\pi/4}$. Here G_1 is independent of time and given by

$$G_1(x; x') = \ln(rr_1) + 2\pi e^{k_0(z+z')} \sin k_0(x - x') + 2 ⨍_0^\infty \frac{\cos k(x - x')}{k - k_0} e^{k(z+z')}\, dk, \tag{3.5}$$

where $k_0 \equiv 4\kappa$ and Cauchy principle-value integral is indicated. Note that G_1 is the well-known steady forward-speed (Kelvin) Green function (Wehausen & Laitone 1960).

3.1.2. *Case II: $\sigma(\tau) = 0$ for $\tau \leqslant t_0$, $\sigma(\tau) = \tau^{-1/2} e^{-i\omega_c \tau}$ for $t_0 < \tau < t$*

If a source is brought into existence at $t = t_0$ with a varying strength $t^{-1/2} e^{-i\omega_c t}$, $\omega_c = g/4U$, the resulting potential $\Psi(x, t; x', \sigma) \equiv \Psi_2(x, t; x')$ at large time t can be expanded as

$$\Psi_2 = \frac{e^{-i\omega_c t}}{t^{1/2}} \left[(C_2 t^{1/2} + \bar{C}_2) e^{\kappa\varphi} + G_2(x; x')\right] + \hat{C}_2 \frac{e^{i\omega_c t}}{t^{1/2}} e^{\kappa\varphi^*} + O\left(\frac{1}{t}, \frac{e^{-i\omega_c t}}{t}\right), \tag{3.6}$$

as $t \to \infty$, where $*$ denotes the complex conjugate. Here the constants C_2, $\bar{C}_2$, and $\hat{C}_2$ are respectively given by

$$C_2 = e^{-i\pi/4} (k_0 U)^{1/2} \sum_{n=0}^{\infty} \frac{\Upsilon(n + \frac{1}{2})}{(2n + 1) n!}, \qquad \bar{C}_2 = -2 e^{-i\pi/4} (2\pi\lambda_0)^{1/2}, \tag{3.7}$$

and

$$\hat{C}_2 = e^{i\pi/4} (2\pi)^{1/2} \int_{\lambda_0}^{\infty} (\cos\lambda - i \sin\lambda) \frac{d\lambda}{\lambda^{1/2}}, \tag{3.8}$$

where Υ is the gamma function and $\lambda_0 \equiv k_0 U t_0 / 2$. The time-independent function G_2 has the following form:

$$G_2(x; x') = \ln\left(\frac{r}{r_1}\right) + i\sqrt{2}\pi \left(e^{m_1^2 \varphi^*} - e^{m_2^2 \varphi^*}\right) + m_0 \left\{ -2 \int_0^\infty \frac{e^{m^2 \varphi}}{(m + m_0/2)^2}\, dm \right.$$
$$\left. - 2 ⨍_0^\infty \frac{e^{m^2 \varphi^*}}{(m + m_1)(m - m_2)}\, dm + \int_L \frac{e^{m^2 \varphi^*}}{(m - m_1)(m + m_2)}\, dm \right\}, \tag{3.9}$$

where $m_0 = k_0^{1/2}$, $m_{1,2} = m_0(\sqrt{2} \pm 1)/2$, and $⨍$ indicates that the path of integration goes below the pole. The contour L extends from $-\infty$ to $+\infty$ in the complex m-plane and is indented to pass below the pole at $m = m_1$ and above the pole at $m = -m_2$.

We remark that in the present case of a source strength oscillating at the critical frequency ω_c, the potential is $O(t^{1/2})$ larger than the source itself, as shown in (3.6).

3.1.3. *Case III:* $\sigma(\tau) = q(\tau) \neq 0$ *for* $\tau < t_0$*;* $\sigma(\tau) = 0$ *for* $\tau \geqslant t_0$

If the source of strength $q(\tau)$ vanishes after a finite time t_0, the resulting potential $\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma) \equiv \Psi_3(\boldsymbol{x}, t; \boldsymbol{x}', q)$ has, at large time t, the form

$$\Psi_3 = C_3 \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}} \mathrm{e}^{\kappa\psi} + \text{c.c.} + O\left(\frac{1}{t}, \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t \to \infty\,, \tag{3.10}$$

where the constant C_3 is given by

$$C_3 = (\pi k_0 U)^{1/2} \mathrm{e}^{-\mathrm{i}\pi/4} \int_{-\infty}^{t_0} q(\tau) \mathrm{e}^{\mathrm{i}\omega_c \tau}\, \mathrm{d}\tau\,. \tag{3.11}$$

3.2. *Three-dimensional sources*

In three dimensions, the solution for Ψ can be expressed as

$$\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma) = \sigma(t)\left(\frac{1}{R} - \frac{1}{R_1}\right) - \frac{\mathrm{i}}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^{\infty} (gk)^{1/2} \mathrm{e}^{k[(z+z')+\mathrm{i}\varpi(\theta)]}$$
$$\times \left\{ \mathrm{e}^{\mathrm{i}t(gk)^{1/2}} \int_{-\infty}^{t} \sigma(\tau) \mathrm{e}^{-\mathrm{i}[(gk)^{1/2}+kU\cos\theta]\tau}\, \mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}t(gk)^{1/2}} \int_{-\infty}^{t} \sigma(\tau) \mathrm{e}^{\mathrm{i}[(gk)^{1/2}-kU\cos\theta]\tau}\, \mathrm{d}\tau \right\} \mathrm{d}k + \text{c.c.}, \tag{3.12}$$

where $R^2, R_1^2 = (x - x')^2 + (y - y')^2 + (z \mp z')^2$, $\varpi(\theta) = \xi \cos\theta + (y - y')\sin\theta$, and $\xi = x - x' + Ut$. Reducing the range of integration from $(-\pi, \pi)$ to $(0, \pi/2)$ and changing variable with $k = m^2$, we can rewrite (3.12) as

$$\Psi(\boldsymbol{x}, t; \boldsymbol{x}', \sigma) = \sigma(t)\left(\frac{1}{R} - \frac{1}{R_1}\right)$$
$$-\frac{\mathrm{i}4g^{1/2}}{\pi}\int_0^{\pi/2} \mathrm{d}\theta \int_0^{\infty} \chi(m,\theta)\, \mathrm{e}^{\mathrm{i}mtg^{1/2}}\, \mathrm{d}m \int_{-\infty}^{t} \sigma(\tau)\, \mathrm{e}^{-\mathrm{i}(mg^{1/2}+m^2U\cos\theta)\tau}\, \mathrm{d}\tau + \text{c.c.}$$
$$+\frac{\mathrm{i}2g^{1/2}}{\pi}\int_0^{\pi/2} \mathrm{d}\theta \int_{-\infty}^{\infty} \chi(m,\theta)\, \mathrm{e}^{-\mathrm{i}mtg^{1/2}}\, \mathrm{d}m \int_{-\infty}^{t} \sigma(\tau)\, \mathrm{e}^{\mathrm{i}(mg^{1/2}-m^2U\cos\theta)\tau}\, \mathrm{d}\tau + \text{c.c.}, \tag{3.13}$$

where $\chi(m, \theta) = m^2 \cos[m^2(y - y')\sin\theta] \mathrm{e}^{m^2(z+z')+\mathrm{i}m^2\xi\cos\theta}$.

Similar to that for a two-dimensional source, the large-time expansions of Ψ can again be obtained from (3.13) using the method of steepest descent. Large-time asymptotic results analogous to those in §3.1 can again be obtained for three specific source variations $\sigma(t)$ which are required in the later analysis. The derivations and results for the three-dimensional cases are given in Appendix B.

4. Time-domain analysis

In this section, we obtain the time dependence of the velocity potential $\Phi(\boldsymbol{x}, t)$ for the body by solving the initial-boundary-value problem in the time domain. In a source formulation, we construct Φ in terms of a time-dependent wave-source distribution over the body. Upon satisfying the requisite body boundary condition, the source strength at any point on the body at any time t is governed by an integral equation over the body which involves convolution over all time $\tau \leqslant t$ of the body sources.

The key part of the asymptotic analysis is to obtain the large-time solution behaviour due to the convolution integral. To facilitate this, it is useful to express the body sources as a sum of separate components based on their time dependence: (i) a time-independent component ($\bar{\sigma}$); (ii) an unsteady component ($\check{\sigma}$) which is zero after some time $t > t_0$; and (iii) an unsteady component ($\hat{\sigma}$) which is non-zero only for $t > t_0$. Much of the analysis involves working out the large-time asymptotic behaviours of the single-source solutions Ψ corresponding to the convolution of these source components and are summarized in §3. Making use of Ψ, one then obtains from the body integral equation the large-time dependence of the source distribution as well as the velocity potential.

For the sake of clarity, we present the analysis in detail only for the case of a two-dimensional body. The extension to three dimensions follows in a straightforward manner and is outlined at the end of this section.

We construct the solution for the velocity potential in terms of a source distribution on the two-dimensional body (Lunde 1951; Ursell 1980):

$$\Phi(\boldsymbol{x},t) = \int_{S_B} \Psi(\boldsymbol{x},t;\boldsymbol{x}',\sigma(\boldsymbol{x}',\tau))\,\mathrm{d}s' - \ell[\Psi(\boldsymbol{x},t;\boldsymbol{x}'_-,\sigma_-(\tau)) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_+,\sigma_+(\tau))]\,, \quad (4.1)$$

where $\ell \equiv U^2/g$, $\sigma(\boldsymbol{x}',\tau)$ represents the source distribution on the body, and, for a surface-intersecting body, $\sigma_\pm(\tau)$ represent the source strengths at the two intersection points, $\boldsymbol{x}'_\pm=(x'_\pm,0)$. Note that for the surface-intersecting case, the form of (4.1) is chosen so that in the steady-state limit, it takes the 'least-singular' form given by Ursell (1980) for the Kelvin–Neumann problem. It is known that other supplementary conditions can be applied for the uniqueness of that problem (see e.g. Suzuki 1981). As is evident from the analysis that follows, such conditions do not affect the present results for the long-time behaviour.

Decomposing the source strength into steady and unsteady components, we have

$$\sigma(\boldsymbol{x}',\tau) = \bar{\sigma}(\boldsymbol{x}') + \tilde{\sigma}(\boldsymbol{x}',\tau) \quad (4.2)$$

in which the time-dependent part $\tilde{\sigma}$ is generally expected to vanish at large time. After substituting (4.2) into (4.1), the velocity potential Φ can be expressed as

$$\Phi(\boldsymbol{x},t) = \int_{S_B} \bar{\sigma}(\boldsymbol{x}')\Psi_1(\boldsymbol{x},t;\boldsymbol{x}')\,\mathrm{d}s' + \int_{S_B} \Psi(\boldsymbol{x},t;\boldsymbol{x}',\tilde{\sigma}(\boldsymbol{x}',\tau))\,\mathrm{d}s'$$
$$-\ell[\bar{\sigma}_-\Psi_1(\boldsymbol{x},t;\boldsymbol{x}'_-) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_-,\tilde{\sigma}_-) + \bar{\sigma}_+\Psi_1(\boldsymbol{x},t;\boldsymbol{x}'_+) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_+,\tilde{\sigma}_+)]. \quad (4.3)$$

For convenience in the later analysis, we further split $\tilde{\sigma}$ into two parts:

$$\tilde{\sigma}(\boldsymbol{x}',\tau) = \check{\sigma}(\boldsymbol{x}',\tau) + \hat{\sigma}(\boldsymbol{x}',\tau) \quad (4.4)$$

where $\check{\sigma}=H(t_0-\tau)\tilde{\sigma}(\boldsymbol{x}',\tau) \equiv q(\boldsymbol{x}',\tau)$ and $\hat{\sigma}=H(\tau-t_0)\tilde{\sigma}(\boldsymbol{x}',\tau)$, and H is the Heaviside function. The velocity potential can now be written in terms of the single-source potentials of §3 as

$$\Phi(\boldsymbol{x},t) = \int_{S_B} \bar{\sigma}(\boldsymbol{x}')\Psi_1(\boldsymbol{x},t;\boldsymbol{x}')\,\mathrm{d}s' + \int_{S_B} \Psi_3(\boldsymbol{x},t;\boldsymbol{x}',q)\,\mathrm{d}s' + \int_{S_B} \Psi(\boldsymbol{x},t;\boldsymbol{x}',\hat{\sigma}(\boldsymbol{x}',\tau))\,\mathrm{d}s'$$
$$-\ell[\bar{\sigma}_-\Psi_1(\boldsymbol{x},t;\boldsymbol{x}'_-) + \Psi_3(\boldsymbol{x},t;\boldsymbol{x}'_-,q_-) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_-,\hat{\sigma}_-)$$
$$+\bar{\sigma}_+\Psi_1(\boldsymbol{x},t;\boldsymbol{x}'_+) + \Psi_3(\boldsymbol{x},t;\boldsymbol{x}'_+,q_+) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_+,\hat{\sigma}_+)]. \quad (4.5)$$

To determine the time dependence of Φ, we first substitute the large-time expansions of $\Psi_1(\boldsymbol{x},t;\boldsymbol{x}')$ and $\Psi_3(\boldsymbol{x},t;\boldsymbol{x}',q)$ into (4.5) to obtain

$$\begin{aligned}\Phi(\boldsymbol{x},t) = &\int_{S_B} \bar{\sigma}(\boldsymbol{x}')G_1(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' + \int_{S_B} \Psi(\boldsymbol{x},t;\boldsymbol{x}',\hat{\sigma}(\boldsymbol{x}',\tau))\,\mathrm{d}s' \\ &-\ell[\bar{\sigma}_- G_1(\boldsymbol{x};\boldsymbol{x}'_-) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_-,\hat{\sigma}_-) + \bar{\sigma}_+ G_1(\boldsymbol{x};\boldsymbol{x}'_+) + \Psi(\boldsymbol{x},t;\boldsymbol{x}'_+,\hat{\sigma}_+)] \\ &+\frac{\alpha \mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\mathrm{e}^{\kappa(\mathrm{i}x+z)} + \mathrm{c.c.} + O\left(\frac{1}{t},\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t\to\infty\,, \qquad (4.6)\end{aligned}$$

where the constant α is given by

$$\begin{aligned}\alpha = &\int_{S_B} \left[C_1\bar{\sigma}(\boldsymbol{x}') + C_3(\boldsymbol{x}')\right] \mathrm{e}^{\kappa(-\mathrm{i}x'+z')}\,\mathrm{d}s' \\ &-\ell\left[C_1\bar{\sigma}_- + C_3(\boldsymbol{x}'_-)\right]\mathrm{e}^{-\mathrm{i}\kappa x'_-} - \ell\left[C_1\bar{\sigma}_+ + C_3(\boldsymbol{x}'_+)\right]\mathrm{e}^{-\mathrm{i}\kappa x'_+}\,. \qquad (4.7)\end{aligned}$$

It is clear that the large-time transient behaviour of Φ is determined by the time dependence of the unsteady source distribution $\hat{\sigma}$ and the value of α. Note that (4.6) satisfies all the conditions of the initial-boundary-value problem except that on the body. In the following, we carry out an analysis to determine $\hat{\sigma}$ and α from the integral equation that results from satisfying the body boundary condition. For simplicity, we restrict the analysis for now to the case of a submerged body for which the point sources at the surface intersection are absent. The case of a surface-piercing body with the intersection point sources turns out to be not fundamentally different in the present context and will be discussed separately at the end of this section.

Upon imposing the body boundary condition (2.3) to (4.6), we obtain an integral equation for the unknown source strength:

$$\begin{aligned}\pi[\bar{\sigma}(\boldsymbol{x}) + \hat{\sigma}(\boldsymbol{x},t)] + &⨍_{S_B} \bar{\sigma}(\boldsymbol{x}')\frac{\partial}{\partial n}G_1(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' + ⨍_{S_B} \frac{\partial}{\partial n}\Psi(\boldsymbol{x},t;\boldsymbol{x}',\hat{\sigma}(\boldsymbol{x}',\tau))\,\mathrm{d}s' \\ &+ \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\alpha\kappa(\mathrm{i}n_x + n_z)\mathrm{e}^{\kappa(\mathrm{i}x+z)} + \mathrm{c.c.} = Un_x + O\left(\frac{1}{t},\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \quad t\to\infty, \quad (4.8)\end{aligned}$$

for all $\boldsymbol{x}$ on the body. According to the definition (4.2), (4.8) can be separated into steady and time-dependent components:

$$\pi\bar{\sigma}(\boldsymbol{x}) + ⨍_{S_B} \bar{\sigma}(\boldsymbol{x}')\frac{\partial}{\partial n}G_1(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' = Un_x\,, \qquad (4.9)$$

and

$$\begin{aligned}\pi\hat{\sigma}(\boldsymbol{x},t) + &⨍_{S_B} \frac{\partial}{\partial n}\Psi(\boldsymbol{x},t;\boldsymbol{x}',\hat{\sigma}(\boldsymbol{x}',\tau))\,\mathrm{d}s' \\ &+ \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\alpha\kappa(\mathrm{i}n_x + n_z)\mathrm{e}^{\kappa(\mathrm{i}x+z)} + \mathrm{c.c.} = O\left(\frac{1}{t},\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t\to\infty\,. \quad (4.10)\end{aligned}$$

Equation (4.9) governs the steady source $\bar{\sigma}(\boldsymbol{x})$ and is associated with the steady Kelvin–Neumann problem; it has been the subject of a large number of studies (e.g. Wehausen 1973). Since the primary focus of this paper is on the unsteady solution, we do not consider (4.9) any further.

Equation (4.10) governs the asymptotic behaviour of the unsteady source $\hat{\sigma}$. From the asymptotic expansions of the single-source pontentials Ψ_1, Ψ_2 and Ψ_3, it is seen that for any time dependence of $\hat{\sigma}$, the associated potential Ψ possesses oscillatory behaviour $\mathrm{e}^{-\mathrm{i}\omega_c t}$ at large time. From (4.10), it follows that the leading-order solution of $\hat{\sigma}$ must also behave like $\mathrm{e}^{-\mathrm{i}\omega_c t}$. In order for the focing term in (4.10) to be formally balanced by terms containing $\hat{\sigma}$, the oscillation must decay at the rate of $t^{-1/2}$. It follows that the asymptotic expansion for $\hat{\sigma}$ must have the form

$$\hat{\sigma}(\boldsymbol{x},t) = \mathrm{Re}\left\{\mu(\boldsymbol{x})\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\right\} + O\left(\frac{1}{t}, \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t \to \infty, \tag{4.11}$$

where μ is complex. Upon substituting (4.11) into (4.6) and applying the large-time expansion of $\Psi_2(\boldsymbol{x},t;\boldsymbol{x}')$, we obtain for the unsteady part of the potential:

$$\begin{aligned}\Phi(\boldsymbol{x},t) - \int_{S_B} \bar{\sigma}(\boldsymbol{x}')G_1(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' &= \frac{\mathrm{e}^{\mathrm{i}\omega_c t}}{t^{1/2}}\beta_1 \mathrm{e}^{\kappa(-\mathrm{i}x+z)} \\ &+ \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\left[\beta_2 \mathrm{e}^{\kappa(\mathrm{i}x+z)} + \int_{S_B} \mu(\boldsymbol{x}')G_2(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s'\right] + O\left(\frac{1}{t}, \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t}\right),\end{aligned} \tag{4.12}$$

as $t \to \infty$, where the real part of the right-hand side is implied. Here β_1 and β_2 are the Kochin functions defined respectively by

$$\beta_1 = \alpha^* + \hat{C}_2 \int_{S_B} \mu(\boldsymbol{x}')\mathrm{e}^{\kappa(\mathrm{i}x'+z')}\,\mathrm{d}s' + \gamma_1\,, \tag{4.13}$$

and

$$\beta_2 = \alpha + (C_2 t^{1/2} + \bar{C}_2) \int_{S_B} \mu(\boldsymbol{x}')\mathrm{e}^{\kappa(-\mathrm{i}x'+z')}\,\mathrm{d}s' + \gamma_2\,, \tag{4.14}$$

where the constants γ_1 and γ_2 represent effects of the source distributions in (the last term in) (4.11) which are higher order in t. For submerged bodies, the constant α is given by (4.7) without the point-source terms at the surface intersections.

From (4.12), it is clear that the leading-order time dependence of Φ is determined by the Kochin functions β_1 and β_2 as well as the source distribution $\mu(\boldsymbol{x})$. We remark that γ_1 and γ_2 should be included in (4.13) and (4.14), because in the large-time expansion of $\Psi(\boldsymbol{x},t;\boldsymbol{x}',\sigma(\tau))$ with $\sigma(t) < O(t^{-1/2})$, there always exists a term decaying like $O(t^{-1/2}\mathrm{e}^{-\mathrm{i}\omega_c t})$ as $t \to \infty$.

In order to determine the magnitudes of β_1, β_2, and $\mu(\boldsymbol{x})$, we substitute (4.11) back into (4.10) and apply the large-time expansion of $\Psi_2(\boldsymbol{x},t;\boldsymbol{x}')$ to obtain

$$\begin{aligned}&\frac{\mathrm{e}^{\mathrm{i}\omega_c t}}{t^{1/2}}\beta_1\kappa(-\mathrm{i}n_x + n_z)\mathrm{e}^{\kappa(-\mathrm{i}x+z)} + \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\left[\pi\mu(\boldsymbol{x}) + \int\!\!\!\!\!\!-_{S_B} \mu(\boldsymbol{x}')\frac{\partial}{\partial n}G_2(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s'\right. \\ &\qquad \left. + \beta_2\kappa(\mathrm{i}n_x + n_z)\mathrm{e}^{\kappa(\mathrm{i}x+z)}\right] = O\left(\frac{1}{t}, \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t}\right), \qquad t \to \infty.\end{aligned} \tag{4.15}$$

Upon identifying the coefficients of the separate time dependencies in (4.15), we have

$$\beta_1\kappa(-\mathrm{i}n_x + n_z)\mathrm{e}^{\kappa(-\mathrm{i}x+z)} = 0\,, \tag{4.16}$$

and

$$\pi\mu(\boldsymbol{x}) + \int\!\!\!\!\!\!-_{S_B} \mu(\boldsymbol{x}')\frac{\partial}{\partial n}G_2(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' = -\beta_2\kappa(\mathrm{i}n_x + n_z)\mathrm{e}^{\kappa(\mathrm{i}x+z)}\,. \tag{4.17}$$

We observe that (4.16) must be satisfied for all $\boldsymbol{x}$ on the body, and so the Kochin function $\beta_1 \equiv 0$.

We are then left with the coupled integral equations (4.14) and (4.17) for the unknowns μ and β_2. To decouple the two equations, we follow the procedure employed in LY. First, we rewrite (4.17) as

$$\mu(x) = -\frac{1}{\pi} \int_{S_B} \mu(x') \frac{\partial}{\partial n} G_2(x; x')\, ds' - \frac{\beta_2 \kappa}{\pi} (in_x + n_z) e^{\kappa(ix+z)} . \tag{4.18}$$

We then substitute (4.18) into (4.14) and solve for the Kochin function β_2 to obtain

$$\beta_2 = \frac{\pi(\alpha + \gamma_2)}{\pi + (C_2 t^{1/2} + \bar{C}_2)\kappa \Gamma} - \frac{1}{\kappa} \int_{S_B} \mu(x') P(x')\, ds' , \qquad t \to \infty , \tag{4.19}$$

where the kernel P is

$$P(x') = \frac{1}{\Gamma} \int_{S_B} \frac{\partial}{\partial n} G_2(x; x') e^{\kappa(-ix+z)}\, ds , \tag{4.20}$$

with the constant Γ given by

$$\Gamma = \int_{S_B} (in_x + n_z) e^{2\kappa z}\, ds , \tag{4.21}$$

where $\kappa = g/4U^2 = 4\omega_c^2/g$.

After substituting (4.19) back into (4.17), we obtain a new integral equation for μ:

$$\pi\mu(x) + \int_{S_B} \mu(x') \left[\frac{\partial}{\partial n} G_2(x; x') + Q(x; x') \right] ds' = \mathscr{R} , \qquad x \in S_B , \tag{4.22}$$

where the forcing term $\mathscr{R}$ is

$$\mathscr{R} \equiv - \left[\frac{\pi(\alpha + \gamma_2)}{\pi + (C_2 t^{1/2} + \bar{C}_2)\kappa \Gamma} \right] \kappa (in_x + n_z) e^{\kappa(ix+z)} , \tag{4.23}$$

and the kernel Q is given by

$$Q(x; x') = (in_x + n_z) e^{\kappa(ix+z)} P(x') . \tag{4.24}$$

According to Fredholm theory (e.g. Ursell 1968), equation (4.22) in general possesses a unique solution except possibly at an enumerable number of discrete values of κ where the Fredholm determinant vanishes. For submerged bodies and sufficiently small U, such irregular frequencies are known to be absent (see Kochin 1937 for the steady problem). Since we are concerned at present with the general κ case, we will disregard the possibility of such irregular frequencies hereafter.

Depending on the value of the geometric parameter Γ, there are two possibilities. If $\Gamma \neq 0$, the forcing $\mathscr{R} \equiv 0$ as $t \to \infty$. From (4.22), we obtain that $\mu(x) \equiv 0$ for $x \in S_B$ and thus

$$\hat{\sigma}(x, t) = O\left(\frac{1}{t}, \frac{e^{-i\omega_c t}}{t^{3/2}} \right), \qquad t \to \infty , \tag{4.25}$$

from (4.11). From (4.17) or (4.19), the Kochin function β_2 must also vanish as $t \to \infty$. According to (4.12), the transient velocity potential thus decays at least like $O(t^{-1}, t^{-1}e^{-i\omega_c t})$ for $t \gg 1$.

The asymptotic behaviour of the potentials due to source distributions of $O(t^{-1}, t^{-3/2}e^{-i\omega_c t})$ on the body can be obtained in a manner identical to that in Appendix A, and the details are not presented. The final result is that the $O(t^{-3/2}e^{-i\omega_c t})$ source produces $O(t^{-1}e^{-i\omega_c t})$ potential, while the $O(t^{-1})$ source does not contribute to it. The resulting potential on the body thus has the asymptote, $t^{-1}e^{-i\omega_c t}\beta e^{\kappa(ix+z)}$ for $t \gg 1$, where β is the Kochin function associated with the $O(t^{-3/2}e^{-i\omega_c t})$ source distribution on the body. Now, at $O(t^{-1}e^{-i\omega_c t})$, there is no forcing and the kinematic condition on the body requires that β=0. The oscillatory unsteady potential must then necessarily be (at least) one order higher, i.e.$O(t^{-3/2}\mathrm{e}^{-\mathrm{i}\omega_c t})$. The final result is that the unsteady part of the total velocity potential in (4.12) must obtain the following asymptotic form:

$$\Phi(\boldsymbol{x},t) - \int_{S_B} \bar{\sigma}(\boldsymbol{x}')G_1(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' = O\left(\frac{1}{t}, \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t \to \infty, \tag{4.26}$$

provided that $\Gamma \neq 0$.

If $\Gamma = 0$, the forcing in (4.22) reduces to $\mathscr{R} = -(\alpha + \gamma_2)\kappa(\mathrm{i}n_x + n_z)\,\mathrm{e}^{\kappa(\mathrm{i}x+z)} \not\equiv 0$, in general. It follows that $\mu \not\equiv 0$ from (4.22), and therefore $\beta_2 \not\equiv 0$ from (4.17). According to (4.11) and (4.12), the unsteady source distribution $\hat{\sigma}$ and velocity potential thus decay like $O(t^{-1/2}\mathrm{e}^{-\mathrm{i}\omega_c t})$ for $t \gg 1$. The asymptotic behaviour of the potential is then identical to that of the impulsively started constant-strength point source (Havelock 1949; equation 3.4).

If the body is surface-piercing, we proceed by including the effect of point sources at the surface intersections in (4.6). After substituting (4.11) into (4.6) and applying the large-time expansion of $\Psi_2(\boldsymbol{x},t;\boldsymbol{x}')$, we obtain

$$\begin{aligned}\Phi(\boldsymbol{x},t) = &\int_{S_B} \bar{\sigma}(\boldsymbol{x}')G_1(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' - \ell[\bar{\sigma}_- G_1(\boldsymbol{x};\boldsymbol{x}'_-) + \bar{\sigma}_+ G_1(\boldsymbol{x};\boldsymbol{x}'_+)] + \frac{\mathrm{e}^{\mathrm{i}\omega_c t}}{t^{1/2}}\beta_1 \mathrm{e}^{\kappa(-\mathrm{i}x+z)} \\ &+\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\left\{\beta_2 \mathrm{e}^{\kappa(\mathrm{i}x+z)} + \int_{S_B}\mu(\boldsymbol{x}')G_2(\boldsymbol{x};\boldsymbol{x}')\,\mathrm{d}s' - \ell\left[\mu_- G_2(\boldsymbol{x};\boldsymbol{x}'_-) + \mu_+ G_2(\boldsymbol{x};\boldsymbol{x}'_+)\right]\right\} \\ &+O\left(\frac{1}{t}, \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t}\right), \qquad t \to \infty,\end{aligned} \tag{4.27}$$

where the Kochin functions β_1 and β_2 for the surface-intersecting body are defined by

$$\beta_1 = \alpha^* + \hat{C}_2\left[\int_{S_B} \mu(\boldsymbol{x}')\mathrm{e}^{\kappa(\mathrm{i}x'+z')}\,\mathrm{d}s' - \ell\left(\mu_- \mathrm{e}^{\mathrm{i}\kappa x'_-} + \mu_+ \mathrm{e}^{\mathrm{i}\kappa x'_+}\right)\right] + \gamma_1, \tag{4.28}$$

and

$$\beta_2 = \alpha + (C_2 t^{1/2} + \bar{C}_2)\left[\int_{S_B} \mu(\boldsymbol{x}')\mathrm{e}^{\kappa(-\mathrm{i}x'+z')}\,\mathrm{d}s' - \ell\left(\mu_- \mathrm{e}^{-\mathrm{i}\kappa x'_-} + \mu_+ \mathrm{e}^{-\mathrm{i}\kappa x'_+}\right)\right] + \gamma_2. \tag{4.29}$$

Following the procedure from (4.15) to (4.22), we can again show that if $\Gamma \neq 0$, the source distribution $\mu(\boldsymbol{x}) \equiv 0$ and the Kochin functions β_1=$\beta_2 \equiv 0$. If Γ=0, $\mu(\boldsymbol{x})$=$O(1)$ and β_2=$O(1)$. Thus the same conclusions as for the submerged body obtain in this case.

The foregoing analysis can be extended to three dimensions in a straightforward way using the three-dimensional transient single-source potentials (Appendix B). The steps are almost identical and are omitted. The transient disturbances associated with the impulsive start of a two- or three-dimensional body vanish in the same way but with different decay rates. Specifically, we obtain that if the three-dimensional body satisfies the geometric condition $\Gamma \neq 0$, the decay rate is $O(t^{-2}, t^{-2}\mathrm{e}^{-\mathrm{i}\omega_c t})$ as $t \to \infty$. If $\Gamma = 0$, the decay rate becomes $O(t^{-1}\mathrm{e}^{-\mathrm{i}\omega_c t})$ for $t \gg 1$, again the same as that for a three-dimensional impulsively started constant-strength source (Wehausen 1964). The geometric constant Γ is formally still given by (4.21) with S_B now representing the submerged surface of the three-dimensional body.

In summary then, we obtain that for a body started abruptly from rest to a constant forward speed U, the associated transient effects vanish for $t \gg 1$ as $O(t^{-1}, t^{-3/2}\mathrm{e}^{-\mathrm{i}\omega_c t})$

in two dimensions and as $O(t^{-2}, t^{-2}e^{-i\omega_c t})$ in three dimensions, provided that the body satisfies the geometric condition $\Gamma \neq 0$. Otherwise, the decay rate is an order of magnitude slower, $\sim O(t^{-1/2}e^{-i\omega_c t})$ and $\sim O(t^{-1}e^{-i\omega_c t})$ in two and three dimensions respectively. These rates are the same as that for a single impulsively started translating constant-strength wave source. Note that the two parameters of the problem – the body geometry and the speed U – are combined in a single governing parameter Γ defined in (4.21). Furthermore, this parameter Γ is the same as that obtained in LY for the frequency-domain problem near the critical frequency ω_c. (That this must be the case is obtained readily from the frequency domain, see §5). The geometric interpretation of the condition $\Gamma \neq 0$ has been expounded in LY and will not be repeated here.

5. Frequency-domain analysis

The result of §4 can be anticipated from the earlier work of LY who performed a frequency-domain analysis of the classical seakeeping problem (a body translating with steady velocity U while undergoing small oscillations at frequency ω) near the critical frequency $\omega = \omega_c$ corresponding to $\tau = U\omega_c/g = \frac{1}{4}$. For this problem, the single-source Green function is singular at ω_c (Haskind 1954; Wehausen & Laitone 1960), a fact directly related to the slow decay in time of the ω_c oscillations in the corresponding impulsively started constant-strength translating source. LY showed that, for a general class of bodies satisfying the geometric condition $\Gamma \neq 0$, a finite solution exists as $\tau \to \frac{1}{4}$. It should then follow that the transient decay of the corresponding time-domain problem should also be more rapid for this class of bodies than that for the Green function. In this section, we apply the Fourier transform to recover the conclusions obtained in the direct time-domain analysis in §4.

We write the time-dependent potential Φ as an inverse Fourier transform from the frequency domain:

$$\Phi(\mathbf{x}, t) = \frac{1}{2\pi} \int_0^\infty \phi(\mathbf{x}, \omega) e^{i\omega t} \, d\omega + \text{c.c.}, \tag{5.1}$$

where ϕ is the Fourier transform of Φ. It is clear that the potential Φ at large time is dominated by the integration in the neighbourhood of the end point $\omega = 0$ and the singularities of ϕ on the positive ω-axis.

To obtain the behaviour of ϕ, we apply the Fourier operator $\int_0^\infty dt \, e^{-i\omega t}$ to the initial-boundary-value problem of Φ in §2. After taking account of initial conditions, we obtain a boundary-value problem for ϕ which is the classical (frequency-domain) seakeeping problem with body boundary condition given by

$$\frac{\partial \phi}{\partial n} = n_x U \left[\pi\delta(\omega) + \frac{1}{i\omega} \right] \qquad \text{on } S_B, \tag{5.2}$$

where $\delta(\omega)$ is the Dirac delta function. Based on (5.2), we can write ϕ as

$$\phi(\mathbf{x}, \omega) = U \left[\pi\delta(\omega) + \frac{1}{i\omega} \right] \phi_0(\mathbf{x}, \omega), \tag{5.3}$$

where ϕ_0 is the solution with the body forcing $\partial\phi_0/\partial n = n_x$.

For the seakeeping problem, it is known that ϕ_0 is regular except at the critical frequency, ω_c, where the (frequency-domain) Green function is singular. For bodies satisfying the geometric condition $\Gamma \neq 0$, LY showed that ϕ_0 is actually bounded at ω_c. Despite this, the integration in (5.1) near ω_c may still dominate the transient behaviour of Φ at large time depending on the smoothness of ϕ_0 at ω_c. Neglecting exponentially small contributions, we can rewrite (5.1) as

$$\Phi(x,t) \sim \frac{U}{2\pi}\left\{\int_0^{\epsilon} + \int_{\omega_c-\epsilon}^{\omega_c+\epsilon}\right\}\left[\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right]\phi_0(x,\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega + \text{c.c.}, \qquad t \gg 1, \quad (5.4)$$

where ϵ is a small number. The integrals in (5.4) can be evaluated upon obtaining the explicit dependence of ϕ_0 on ω.

5.1. *Two-dimensional bodies*

For the behaviour of ϕ_0 at low frequency, we expand the frequency-domain wave-source Green function, $G(x;x',\omega)$, about $\omega = 0$ to obtain

$$G(x;x',\omega) = G_{00}(x;x') + \omega G_{01}(x;x') + O(\omega^2), \qquad \omega \ll 1, \qquad (5.5)$$

where G_{00} and G_{01} are independent of ω. Expressing the potential in a source formulation, it follows directly that ϕ_0 must have a similar expansion:

$$\phi_0(x,\omega) = \phi_{00}(x) + \omega\phi_{01}(x) + O(\omega^2), \qquad \omega \ll 1, \qquad (5.6)$$

where again ϕ_{00} and ϕ_{01} are independent of ω.

Near the critical frequency ω_c, LY derived an asymptotic expression for ϕ_0 which can be formally expressed as

$$\phi_0(x,\omega) = \frac{f(x)}{(\omega_c-\omega)^{1/2} + d\omega^2\Gamma} + O((\omega_c-\omega)^{1/2}), \qquad |\omega_c - \omega| \ll 1, \qquad (5.7)$$

where the function f and the constant d are independent of frequency ω. If $\Gamma \neq 0$, ϕ_0 is bounded, but its first derivative (with respect to ω) possesses a square-root singularity at $\omega = \omega_c$. If Γ=0, ϕ_0 itself has a square-root singularity at ω=ω_c.

Upon substituting (5.7) and (5.6) into (5.4), and integrating by parts, we obtain

$$\Phi(x,t) - U\phi_{00}(x)/2 \sim \mathrm{e}^{-\mathrm{i}\omega_c t}\int_{-\epsilon}^{\epsilon}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega^{1/2} + d\omega_c^2\Gamma}\,\mathrm{d}\omega + O\left(t^{-1}, t^{-3/2}\mathrm{e}^{-\mathrm{i}\omega_c t}\right), \qquad t \gg 1. \qquad (5.8)$$

If $\Gamma = 0$, the leading behaviour of the unsteady solution is dominated by the first integral, which can be evaluated to give

$$\Phi(x,t) - U\phi_{00}(x)/2 = O\left(t^{-1/2}\mathrm{e}^{-\mathrm{i}\omega_c t}\right) \qquad \text{for} \qquad \Gamma = 0. \qquad (5.9)$$

If $\Gamma \neq 0$, the first integral in (5.8) is regular and can be evaluated by expanding the integrand in a Taylor series around ω=0. The leading behaviour is obtained to be

$$\Phi(x,t) - U\phi_{00}(x)/2 = O\left(t^{-1}, t^{-3/2}\mathrm{e}^{-\mathrm{i}\omega_c t}\right) \qquad \text{for} \qquad \Gamma \neq 0. \qquad (5.10)$$

Thus the large-time asymptotic behaviour of §4 is exactly recovered.

It remains of interest to understand the manner in which the leading algebraic behaviour changes in the limit $\Gamma \to 0$. This would be relevant for a problem with small but non-zero Γ as in the case, for example, of small forward speed or deep submergence. In this case, the conclusion above and at the end of §4 for $(1 \gg)\ \Gamma \neq 0$ remains formally valid as $t \to \infty$, although a discontinuity in the behaviour apparently exists between this and the case of $\Gamma = 0$. A useful consideration, then, is of the case of $\Gamma \ll 1$ and large but finite $t \gg 1$.

For this case, the integral in (5.8) can be evaluated although special care is needed since the Taylor expansion around $\omega=0$ does not converge in the limit $\Gamma \to 0$. Upon shifting the integration path to the line of steepest descent, changing variable with $\varrho=(\omega t)^{1/2}$ and defining $\mathscr{B} \equiv d^2\omega_c^4\Gamma^2 \sim \Gamma^2$, the integral in (5.8) becomes

$$\int_{-\epsilon}^{\epsilon} \frac{e^{i\omega t}}{\omega^{1/2}+d\omega_c^2\Gamma}\,d\omega \sim t^{-1/2}\left[\int_0^{\infty} \frac{\varrho e^{-\varrho^2}}{\varrho+(\mathscr{B}t)^{1/2}}\,d\varrho + O(\mathscr{B}t)^{1/2}\right]$$
$$= t^{-1/2}\left[e^{-\mathscr{B}t}\int_{(\mathscr{B}t)^{1/2}}^{\infty} e^{-\varrho^2}\,d\varrho + O(\mathscr{B}t)^{1/2}\right]$$
$$= \tfrac{1}{2}\pi^{1/2}t^{-1/2}e^{-\mathscr{B}t}[1+O(\mathscr{B}t)^{1/2}]\,, \qquad (5.11)$$

for $t \gg 1$ and $\Gamma^2 t \ll 1$.

Thus, the asymptotic behaviour of the unsteady potential can be expressed as

$$\Phi(x,t) - U\phi_{00}(x)/2 = \tfrac{1}{2}\pi^{1/2}t^{-1/2}e^{-\mathscr{B}t}e^{-i\omega_c t}[1+O(\mathscr{B}t)^{1/2}] + O\left(t^{-1}, t^{-3/2}e^{-i\omega_c t}\right)\,, \quad (5.12)$$

for $\Gamma^2 t \ll 1$, $t \gg 1$. As $\Gamma \to 0$ (for finite t), $\mathscr{B}t$ vanishes in the first term which then becomes the leading algebraic term giving the asymptotic behaviour $\sim O(t^{-1/2}e^{-i\omega_c t})$. Hence, we recover the formal result for $\Gamma = 0$. For finite $\mathscr{B} \sim \Gamma^2$, the first term on the right is exponential, and we recover the previous result for $\Gamma \neq 0$. Thus there is no ambiguity in general in the two limits $\Gamma \to 0$ and $t \to \infty$ depending on whether $\mathscr{B}t \to 0$ or not. Similar conclusions are also obtained for three-dimensional bodies (see §§5.2 and 6.2).

5.2. *Three-dimensional bodies*

As in two dimensions, the small-frequency expansion of ϕ_0 can be obtained by expanding the three-dimensional (frequency-domain) wave-source Green function (Wehausen & Laitone 1960) about $\omega = 0$. The result can be expressed in a symbolic form:

$$\phi_0(x,\omega) = \phi_{00}(x) + \omega^2\phi_{01}(x) + \ldots\,, \qquad \omega \ll 1\,. \qquad (5.13)$$

For the asymptotic behaviour of ϕ_0 near ω_c, we follow the procedure outlined in §6 of LY. The solution can be formally expressed as

$$\phi_0(x,\omega) = \frac{F(x)\ln|\omega_c-\omega|}{D+\kappa^2\Gamma\ln|\omega_c-\omega|} + O(|\omega_c-\omega|)\,, \qquad |\omega_c-\omega| \ll 1\,, \qquad (5.14)$$

where the function F and the constant D are independent of ω. Clearly, ϕ_0 is bounded as $\omega \to \omega_c$ if $\Gamma \neq 0$. Otherwise, it has a logarithm singularity at $\omega=\omega_c$.

If $\Gamma=0$, substitution of (5.13) and (5.14) into (5.4) gives

$$\Phi(x,t) - U\phi_{00}(x)/2 \sim e^{-i\omega_c t}\int_0^{\epsilon}(\ln\varrho)\cos\varrho t\,d\varrho + O\left(t^{-2}, t^{-2}e^{-i\omega_c t}\right)\,, \qquad t \gg 1\,. \quad (5.15)$$

Using contour integration, the integral in (5.15) can be shown to be $O(1/t)$ as $t \to \infty$. Thus, for $\Gamma=0$, the transient velocity potential decays like $O(t^{-1}e^{-i\omega_c t})$ for $t \gg 1$.

If $\Gamma \neq 0$, we substitute (5.13) and (5.14) into (5.4) and use integration by parts to obtain

$$\Phi(x,t) - U\phi_{00}(x)/2 \sim t^{-1}e^{-i\omega_c t}\int_0^{\epsilon}\frac{e^{i\varrho t}}{\varrho(D+\kappa^2\Gamma\ln\varrho)^2}\,d\varrho + O\left(t^{-2}, t^{-2}e^{-i\omega_c t}\right) \quad (5.16)$$

for $t \gg 1$. After using the method of steepest descent, we have

$$\Phi(\boldsymbol{x},t) - U\phi_{00}(\boldsymbol{x})/2 \sim t^{-1}\mathrm{e}^{-\mathrm{i}\omega_c t}\int_0^{\epsilon} \frac{\mathrm{e}^{-\varrho t}}{\varrho[D+\kappa^2\Gamma(\mathrm{i}\pi/2+\ln\varrho)]^2}\,\mathrm{d}\varrho$$
$$+\,O\left(t^{-2}, t^{-2}\mathrm{e}^{-\mathrm{i}\omega_c t}\right), \qquad t \gg 1. \tag{5.17}$$

The integral in (5.17) has a form of Ramanujan's integral (Erdélyi 1981, p. 219) and can be shown to vanish exponentially as $t \to \infty$. Equation (5.17) can be formally expressed as

$$\Phi(\boldsymbol{x},t) - U\phi_{00}(\boldsymbol{x})/2 = O\left[\mathrm{e}^{-\mathrm{i}\omega_c t}\left(t^{-1}\mathrm{e}^{-\mathscr{B}t}, t^{-2}\right), t^{-2}\right], \qquad t \gg 1\,, \tag{5.18}$$

where, in this case, $\mathscr{B} \sim \mathrm{e}^{-D/\kappa^2|\Gamma|}$ corresponds to the value of ϱ at which the real part of the denominator in the integrand in (5.17) vanishes. Clearly, $\mathscr{B}$=0 for $\Gamma = 0$ and the leading behaviour of (5.18) reduces to $O(t^{-1}\mathrm{e}^{-\mathrm{i}\omega_c t})$. For $\Gamma \neq 0$, $\mathscr{B}$ is positive and increases monotonically with $|\Gamma|$, the decaying exponential in (5.18) vanishes as $t \to \infty$, and the unsteady potential decays like $O(t^{-2}, t^{-2}\mathrm{e}^{-\mathrm{i}\omega_c t})$. For large but finite t, the first term on the right-hand side of (5.18) becomes the leading algebraic term if $\Gamma \to 0$ ($\mathscr{B}t \to 0$), and the solution decays only as $O(t^{-1}\mathrm{e}^{-\mathrm{i}\omega_c t})$. In any case, the time-domain analysis results of §4 are recovered.

6. Numerical confirmation

As we pointed out, the decay rate of transients associated with body accelerations has immediate implications for testing tank experiments as well as time-domain simulations. To illustrate this and to provide numerical confirmation of the analysis results, we perform direct simulations in the time domain to determine the resistance of a body started impulsively from rest to a constant forward speed. Specifically, we consider two different geometries using two independent time-domain simulation programs: (*a*) a submerged two-dimensional circular cylinder using a spectral method; and (*b*) a three-dimensional ship hull employing a transient-wave Green function panel method.

6.1. *Unsteady resistance of a submerged cylinder*

We consider the time-varying resistance $\mathscr{F}(t)$ on a two-dimensional submerged circular cylinder, radius a, centre submergence h, started impulsively from rest to constant forward speed U. The geometric parameter Γ is obtained to be $\Gamma = 2\pi a\mathrm{e}^{-2\kappa h}\mathscr{I}_1(2\kappa a)$, where $\mathscr{I}_1$ is the modified Bessel function. Thus Γ is always positive.

The initial-boundary-value problem is simulated in the time domain from rest. The numerical method we use is a simple extension, to include forward speed, of the spectral method of Liu, Dommermuth & Yue (1992) for wave–body interactions. The free surface and body surface are represented by dipole and source distributions respectively which are written separately in terms of Fourier series (with N_F, N_B modes). The method exhibits exponential convergence with N_F and N_B. In addition, with the use of fast-Fourier transform, the computational effort is only linearly proportional to N_F (typically $N_B \ll N_F$) so that in practice a large value of N_F (say up to $O(10^3)$) can be used to obtain an extremely high accuracy (Liu *et al.* 1992).

Figure 1 shows the comparison between the numerical result and the fitted asymptotic solution based on (4.26) for the unsteady portion of the resistance, $\mathscr{F}(t) - \overline{\mathscr{F}}$, for Froude number $F_r = U/(ga)^{1/2} = 1$ and submergence $h/a = 2$. The direct

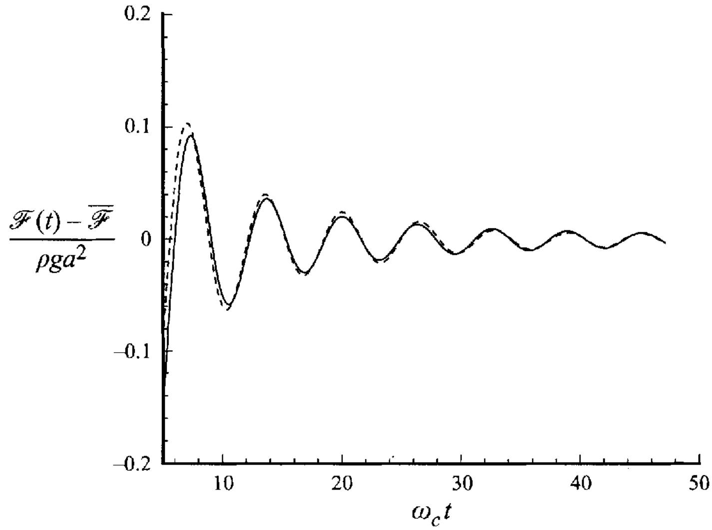

FIGURE 1. Comparison between the asymptotic prediction (6.1) (——) and direct time-domain simulation result (- - -) for the unsteady wave resistance on a submerged circular cylinder, radius a, centre submergence h, accelerated abruptly from rest to a speed U. The parameters of the problem are $F_r = U/(ga)^{1/2} = 1$ and $h/a = 2$.

time-domain simulation uses N_B=64, N_F=2048, in a computational domain of length $64L_0$ where $L_0 = 2\pi U^2/g$ is the wavelength of the steady wave. With these and the time integration (fourth-order Runge–Kutta) parameters chosen, the result for the resistance is converged to at least 4 significant decimals. The fitted theoretical solution corresponding to (4.26) is given by

$$\frac{\mathscr{F}(t)-\overline{\mathscr{F}}}{\rho g a^2} = \frac{a_1}{\omega_c t} + \frac{a_2}{(\omega_c t)^{3/2}} \cos(\omega_c t + a_3) \, , \tag{6.1}$$

with $a_1 \approx -0.033$ and $a_2 \approx 1.687$. The comparison between the theoretical asymptotic prediction and direct numerical computations in figure 1 is excellent, and the results are graphically indistinguishable for $\omega_c t >\sim 30$.

As a further confirmation, we check the predicted asymptotic behaviour for the source strength (4.25) against direct simulation. Figure 2 shows the comparison for the first circumferential Fourier mode of $\hat{\sigma}(\boldsymbol{x}, t)$, which we denote as $\hat{\sigma}_1(t)$, for the same case as figure 1. The theoretical curve for $\hat{\sigma}_1(t)/U$ has the same form as the right-hand side of (6.1) with now $a_1 \approx -0.003$ and $a_2 \approx 0.121$. The curves are very similar to those of figure 1 and the comparison is again excellent.

6.2. *Resistance of a ship started from rest*

As an example of a three-dimensional surface-piercing body, we consider the unsteady resistance $\mathscr{F}(t)$ of a Wigley hull (Gerritsma 1988), length a, beam b, and draught h. The half-beam of the hull is given by $y/b = \pm(1 - 4x^2/a^2)(1 - z^2/h^2)/2$. For this mathematical hull, we calculate $\Gamma = ab[(1 + 2\kappa h)e^{-2\kappa h} - 1]/3(\kappa h)^2$. For the numerical simulation, we use a code based on a time-domain three-dimensional transient-wave Green function panel method of Lin & Yue (1990). The method employs a piecewise linear quadrilateral panel discretization of the hull surface with constant source strength over each panel.

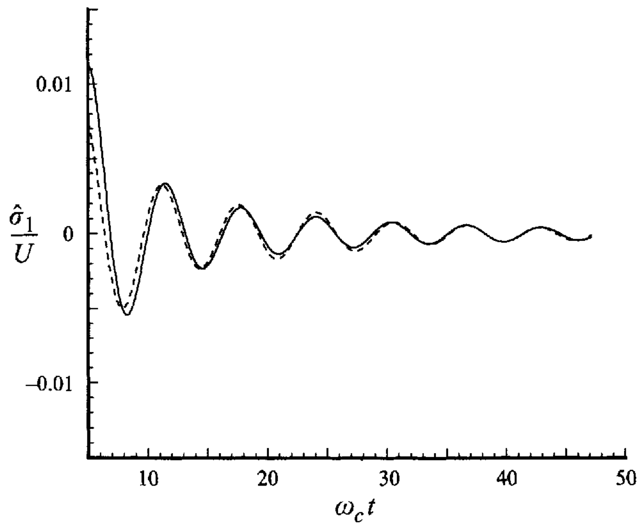

FIGURE 2. Comparison between the asymptotic prediction (cf. (6.1)) (——) and direct time-domain simulation result (- - -) for the first circumferential mode of the unsteady source distribution on a submerged circular cylinder, radius a, centre submergence h, accelerated abruptly from rest to a speed U. The parameters of the problem are $F_r = U/(ga)^{1/2} = 1$ and $h/a = 2$.

Figure 3 plots the unsteady part of the resistance for the case of b/a=0.1, h/a=0.0625, and Froude number F_r=$U/(ga)^{1/2}$=0.125. The time-domain simulation uses $N_B = 434$ panels on the body and 80 (trapezoidal integration) time steps per critical wave period $T_c = 2\pi/\omega_c = 8\pi U/g$. The numerical scheme achieves only algebraic (second-order) convergence with N_B and the displayed result is converged to $O(10^{-2})$. The fitted asymptotic prediction corresponding to (5.18) is given in this case by

$$\frac{\mathscr{F}(t) - \overline{\mathscr{F}}}{\rho gabh} = \frac{a_1}{(\omega_c t)^2} + \frac{a_2}{(\omega_c t)^2} \cos(\omega_c t + a_3) , \tag{6.2}$$

with $a_1 \approx 6.1 \times 10^{-4}$ and $a_2 \approx 0.02$. The agreement between the analytic and numerical results is again very good, and confirms the $O(t^{-2})$ approach to steady-state resistance.

Finally, we remark that in the large-time asymptotic analysis of §4, exponentially small terms are all neglected. The results are then formally valid for $t \to \infty$. The time-dependent resistance in general also contains ω_c oscillatory terms with exponential time-dependent amplitudes of the form $t^{-\mathscr{A}} \mathrm{e}^{-\mathscr{B}t}$ (Maskell & Ursell 1970; Newman 1985), where $\mathscr{A} = \frac{1}{2}$ (1) for two (three) dimensions. These amplitudes are an order of magnitude slower than those for the oscillations in (6.1), (6.2). In practice, the value of $\mathscr{B}$ may be quite small, and the resistance is dominated by the $t^{-\mathscr{A}}$ behaviour for fairly large finite time $t < T$ given by $T\mathrm{e}^{-\mathscr{B}T} = O(1)$. In Fourier analyses (§5), we obtain that $\mathscr{B} \sim (\kappa\Gamma)^2$ in two dimensions and $\mathscr{B}$ is a monotonic function of Γ in three dimensions where $\mathscr{B}$ vanishes for Γ=0 and increases with increasing absolute value of the dimensionless parameter $\mathscr{K} \equiv \kappa^2\Gamma$.

For the Wigley hull, we calculate that

$$|\mathscr{K}| = ab[1 - (1 + h/2aF_r^2)\mathrm{e}^{-h/2aF_r^2}]/3h^2 .$$

Thus, $|\mathscr{K}|$ has the maximum value of $ab/3h^2$ at F_r=0 and decreases monotonically with increasing F_r. For $F_r \gg 1$, $|\mathscr{K}|$ approaches zero as F_r^{-4}. To illustrate the

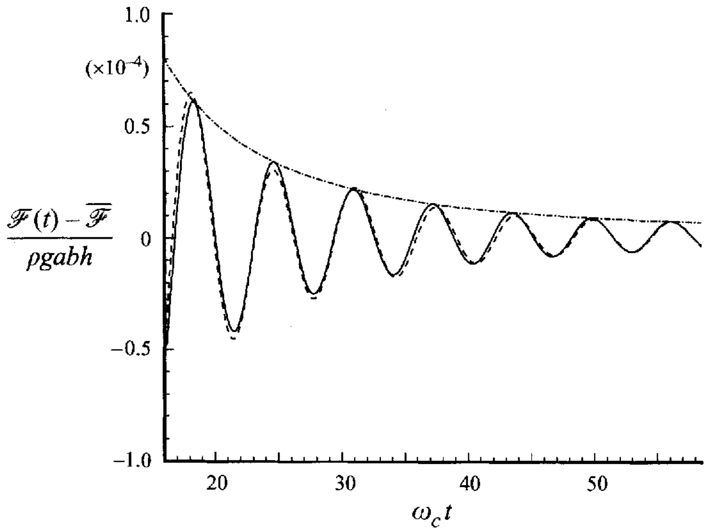

FIGURE 3. Comparison between the asymptotic prediction (6.2) (——) and direct time-domain simulation result (- - -) for the unsteady wave resistance on a Wigley hull, length a, beam b, draught h, accelerated abruptly from rest to a speed U. The parameters of the problem are $F_r = U/(ga)^{1/2}$=0.125, b/a=0.1, and h/a=0.0625. For reference, the envelope of (6.2) (— · —), given by the asymptote $\sim t^{-2}$, is also plotted.

exponential time-dependent oscillatory behaviour, we perform the simulation for a different speed given by F_r=0.3 (with a reduction in the previous value of $|\mathscr{K}|$ for figure 3 by a factor of about 8). The result for the unsteady resistance is shown in figure 4. In this case, an asymptotic expression of the form (6.2) will not match the numerical time history which is dominated by behaviour of the form

$$\frac{\mathscr{F}(t)-\overline{\mathscr{F}}}{\rho gabh} = \frac{a_1}{\omega_c t} e^{-\mathscr{B}t} \cos(\omega_c t + a_3) . \tag{6.3}$$

A fitted curve based on (6.3) with $a_1 \approx 2.4 \times 10^{-3}$ and $\mathscr{B} \approx 2.8 \times 10^{-3}\omega_c$ is plotted in figure 4 which compares quite well with the numerical solution. In this case, we find that the algebraic decay rate of (6.2) becomes dominant only after $t > O(100T_c)$.

7. Summary

We study the decay behaviour of wave transients associated with the initial acceleration of a body from rest to a constant forward speed. The canonical problem of an impulsive start (step function velocity) is considered although the results are applicable to arbitrary (finite-duration) accelerations. We write an initial-boundary-value problem for the velocity potential which is represented as a distribution of transient wave sources on the body surface. The source strength distribution, which is time dependent and *a priori* unknown, is governed by an integral equation on the body in order to satisfy the body boundary condition. We perform an asymptotic analysis of this integral equation to obtain the large-time behaviour of the source strength and thus the total solution.

The time-domain analysis shows that for a general class of bodies satisfying a simple geometric condition $\Gamma \neq 0$, the unsteady part of the solution decays like $O(t^{-1}, t^{-3/2}e^{-i\omega_c t})$ and $O(t^{-2}, t^{-2}e^{-i\omega_c t})$ as $t \to \infty$ respectively for two- and three-

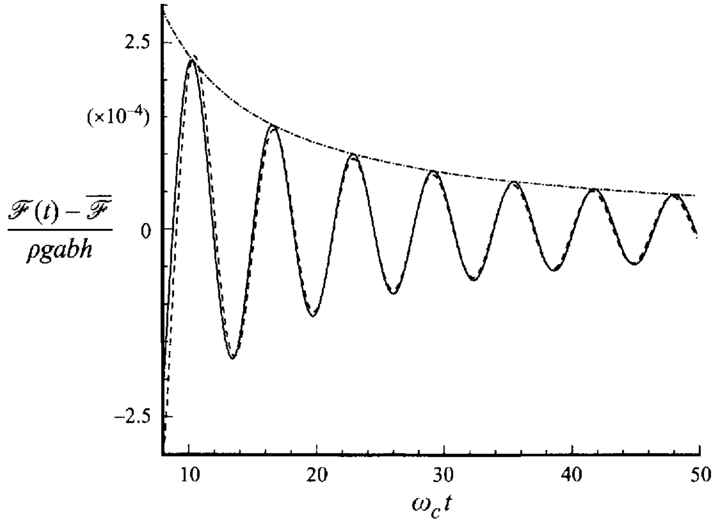

FIGURE 4. Comparison between the asymptotic prediction (6.3) (——) and direct time-domain simulation result (- - -) for the unsteady wave resistance on a Wigley hull, length a, beam b, draught h, accelerated abruptly from rest to a speed U. The parameters of the problem are $F_r = U/(ga)^{1/2} = 0.3$, $b/a = 0.1$, and $h/a = 0.0625$. For reference, the envelope of (6.3) (— · —), given by the asymptote $\sim t^{-1}\mathrm{e}^{-0.0028\omega_c t}$, is also plotted.

dimensional bodies, where $\omega_c = g/4U$ is the critical frequency. When $\Gamma = 0$, for which the single transient translating source is a special case, the decay rate of the ω_c oscillation is an order of magnitude slower and the solution behaves like $O(t^{-1/2}\mathrm{e}^{-\mathrm{i}\omega_c t})$ and $O(t^{-1}\mathrm{e}^{-\mathrm{i}\omega_c t})$ in two and three dimensions respectively. These results are also recovered from the frequency domain after making use of the known behaviour of the (frequency-domain) seakeeping solution near the critical frequency (LY).

The case of $0 \neq \Gamma \ll 1$ (for example a deeply submerged body) is of some interest since there is an apparent discontinuity in the leading solution behaviour for $\Gamma =, \neq 0$ as $t \to \infty$. Further analyses (and numerical results) for $\Gamma \to 0$ but finite $t \gg 1$ reveal the role of a time algebraic-exponential decaying term of the form $t^{-\mathscr{A}} e^{-\mathscr{B}(\Gamma)t} \mathrm{e}^{-\mathrm{i}\omega_c t}$, where $\mathscr{A} = \frac{1}{2}$, 1 for two, three dimensions; $\mathscr{B}(0) = 0$, and $\mathscr{B}(\Gamma)$ is non-zero and increases monotonically with $|\Gamma| > 0$. As $\Gamma \to 0$, $\mathscr{B}t \to 0$ for finite t, and the leading algebraic behaviour is given by $t^{-\mathscr{A}}$. For $\Gamma \neq 0$ and $t \to \infty$, this term is (exponentially) higher order. Thus, the formal conclusions above are recovered.

As an illustration, we perform direct time-domain numerical simulations for two different geometries (a two-dimensional submerged body and a three-dimensional floating body) and compare the results to theoretical predictions. The unsteady behaviour of the solutions completely confirms the asymptotic analyses and illustrates the possibly important practical implications to wave–body studies in the testing tank and in time-domain simulations.

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Appendix A. Large-time expansions of two-dimensional transient single-source potentials

In this appendix, we derive asymptotic expansions, for $|\boldsymbol{x}-\boldsymbol{x}'|/Ut = o(1)$ as $t \to \infty$, of the two-dimensional single-source potential Ψ for three special cases of source strength $\sigma(t)$.

A.1. *Case I: $\sigma(\tau)=0$ for $\tau \leqslant 0$, $\sigma(\tau)=1$ for $0<\tau\leqslant t$*

For $\sigma(\tau)$=1 for $0<\tau\leqslant t$, the integration with respect to τ in (3.3) can be carried out to give

$$\Psi_1(\boldsymbol{x},t;\boldsymbol{x}') = \ln\left(\frac{r}{r_1}\right) + 2 ⨍_0^\infty \left(\frac{1}{k-k_0}-\frac{1}{k}\right) \mathrm{e}^{k(z+z')} \cos k(x-x')\,\mathrm{d}k$$
$$+2\int_0^\infty \left(\frac{1}{m}-\frac{1}{m+m_0}\right)\mathrm{e}^{m^2\psi+\mathrm{i}\Omega_1 t}\,\mathrm{d}m + \text{c.c.}$$
$$+⨍_{-\infty}^\infty \left(\frac{1}{m}-\frac{1}{m-m_0}\right)\mathrm{e}^{m^2\psi+\mathrm{i}\Omega_2 t}\,\mathrm{d}m + \text{c.c.}\,, \qquad \text{(A 1)}$$

where $k_0=g/U^2$, $m_0^2=k_0$, and the first integral is independent of time.

At large time, the second integral in (A 1), which we denote by I_2, is dominated by the integration near m=0. The integration path can be changed to the line of descent $m=\mathrm{i}\varrho/g^{1/2}$. (Although this is not strictly a steepest descent path, the leading time dependence in (A 3) is unaffected). Upon expanding the integrand in Taylor series about ϱ=0, it follows that

$$I_2 \simeq 2\int_0^\epsilon \left[\frac{1}{\varrho}-\frac{\mathrm{i}}{Um_0^2}+O(\varrho)\right]\mathrm{e}^{-\varrho t}\,\mathrm{d}\varrho + \text{c.c.}\,, \qquad t\to\infty\,, \qquad \text{(A 2)}$$

where ϵ represents a small positive number. After changing variable with $\lambda=\varrho t$, we have

$$I_2 \simeq 4\int_0^\infty \frac{\mathrm{e}^{-\lambda}}{\lambda}\,\mathrm{d}\lambda + O\left(\frac{1}{t^2}\right), \qquad t\to\infty\,. \qquad \text{(A 3)}$$

Note that while the integral in (A 3) diverges, its sum with the divergent first integral in (A 1) is convergent (Gel'fand & Shilov 1964) resulting in a constant (non-time-dependent) term.

The last integral, I_3, in (A 1), possesses a saddle point at $m=tg^{1/2}/2\xi$ and two simple poles at m=0 and m_0. Here $\xi=x-x'+Ut$ is always positive. Deforming the path of integration to the lines of steepest descent given by $m_{1,2}= tg^{1/2}/2\xi \pm \varrho\mathrm{e}^{\mathrm{i}\pi/4}$, we obtain

$$I_3 = 2\pi\mathrm{e}^{k_0(z+z')}\sin k_0(x-x')$$
$$+m_0\exp\left[\mathrm{i}\left(\frac{\pi}{4}-\frac{gt^2}{4\xi}\right)\right]\int_0^\infty\left[\frac{\mathrm{e}^{m_1^2(z+z')}}{m_1(m_0-m_1)}+\frac{\mathrm{e}^{m_2^2(z+z')}}{m_2(m_0-m_2)}\right]\mathrm{e}^{-\xi\varrho^2}\,\mathrm{d}\varrho + \text{c.c.} \quad \text{(A 4)}$$

For large ξ, the integral in (A 4) can be evaluated using Laplace's method. By expanding the terms inside the square brackets in Taylor series about ϱ=0, it follows that

$$I_3 \simeq 2\pi\mathrm{e}^{k_0(z+z')}\sin k_0(x-x') + \frac{8m_0\xi^2}{tg^{1/2}(2\xi m_0 - tg^{1/2})}\exp\left[\mathrm{i}\left(\frac{\pi}{4}-\frac{gt^2}{4\xi}\right)+\frac{gt^2}{4\xi^2}(z+z')\right]$$
$$\times\int_0^\epsilon\left[1+O(\varrho^2)\right]\mathrm{e}^{-\xi\varrho^2}\,\mathrm{d}\varrho + \text{c.c.}, \qquad \text{(A 5)}$$

as $\xi \to \infty$. After carrying out the resulting integral, we obtain

$$I_3 = 2\pi e^{k_0(z+z')} \sin k_0(x-x') \\ +4\exp\left[i\left(\frac{\pi}{4}-\frac{gt^2}{4\xi}\right)+\frac{gt^2}{4\xi^2}(z+z')\right]\frac{\xi^{3/2}(\pi/g)^{1/2}}{t(2\xi-Ut)}\left[1+O\left(\frac{1}{\xi}\right)\right]+\text{c.c.}, \quad \xi\to\infty. \tag{A 6}$$

Substituting $\xi = x - x' + Ut$ for (A 6) and applying the condition $|x-x'|/Ut = o(1)$, we can expand (A 6) in a simpler form:

$$I_3 = 2\pi e^{k_0(z+z')} \sin k_0(x-x') \\ +4\left(\frac{\pi}{\kappa U t}\right)^{1/2} e^{\kappa\psi - i\omega_c t + i\pi/4} + \text{c.c.} + O\left(\frac{e^{-i\omega_c t}}{t^{3/2}}\right), \quad t\to\infty, \tag{A 7}$$

where $\kappa = k_0/4$ and $\omega_c = g/4U$.

In summary, the expansion of Ψ_1 can be written as

$$\Psi_1 = G_1(\boldsymbol{x};\boldsymbol{x}') + C_1 \frac{e^{-i\omega_c t}}{t^{1/2}} e^{\kappa\psi} + \text{c.c.} + O\left(\frac{e^{-i\omega_c t}}{t^{3/2}}\right), \quad t\to\infty, \tag{A 8}$$

where the constant $C_1 = 8(\pi U/g)^{1/2} e^{i\pi/4}$ and the time-independent function G_1 is given by

$$G_1(\boldsymbol{x};\boldsymbol{x}') = \ln(rr_1) + 2\pi e^{k_0(z+z')}\sin k_0(x-x') + 2 ⨍_0^\infty \frac{\cos k(x-x')}{k-k_0} e^{k(z+z')}\, dk. \tag{A 9}$$

A.2. *Case II:* $\sigma(\tau) = 0$ *for* $\tau \leqslant t_0$, $\sigma(\tau) = \tau^{-1/2} e^{-i\omega_c\tau}$ *for* $t_0 < \tau < t$

Substituting $\sigma(\tau) = \tau^{-1/2}e^{-i\omega_c\tau}$ for $\tau > t_0$ into (3.3), the resulting velocity potential Ψ_2 for $t > t_0$ can be written as:

$$\Psi_2 = \frac{e^{-i\omega_c t}}{t^{1/2}} \ln\left(\frac{r}{r_1}\right) + i2g^{1/2}\int_0^\infty e^{m^2\psi + i\Omega_1 t}\, dm \int_{t_0}^t \tau^{-1/2} e^{-i(\Omega_1+\kappa U)\tau}\, d\tau \\ - i2g^{1/2}\int_0^\infty e^{m^2\psi^* - i\Omega_1 t}\, dm \int_{t_0}^t \tau^{-1/2} e^{i(\Omega_1-\kappa U)\tau}\, d\tau \\ - ig^{1/2}\int_{-\infty}^\infty e^{m^2\psi + i\Omega_2 t}\, dm \int_{t_0}^t \tau^{-1/2} e^{-i(\Omega_2+\kappa U)\tau}\, d\tau \\ + ig^{1/2}\int_{-\infty}^\infty e^{m^2\psi^* - i\Omega_2 t}\, dm \int_{t_0}^t \tau^{-1/2} e^{i(\Omega_2-\kappa U)\tau}\, d\tau. \tag{A 10}$$

In order to determine the integration with respect to τ in (A 10), we first evaluate the integral

$$\int_{t_0}^t \frac{e^{\pm i\nu\tau}}{\tau^{1/2}}\, d\tau = \frac{W_{1,2}}{\nu^{1/2}} \pm \frac{e^{\pm i\nu t}}{i\nu t^{1/2}} + O\left(\frac{e^{\pm i\nu t}}{t^{3/2}}\right), \quad t\to\infty, \tag{A 11}$$

where ν is a positive constant and $W_{1,2}$ are regular functions of ν and given by

$$W_{1,2}(\nu) = \int_{\nu t_0}^\infty (\cos\lambda \pm i\sin\lambda)\frac{d\lambda}{\lambda^{1/2}}. \tag{A 12}$$

After carrying out the integration with respect to τ, the first line of (A 10) becomes:

$$I_1 = \mathrm{i}2m_0U^{1/2}\int_0^\infty W_2(v_1)\frac{\mathrm{e}^{m^2\psi+\mathrm{i}\Omega_1 t}}{m+m_0/2}\,\mathrm{d}m$$
$$-2m_0\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{t^{1/2}}\int_0^\infty \frac{\mathrm{e}^{m^2\psi}}{(m+m_0/2)^2}\,\mathrm{d}m + O\left(\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t\to\infty, \qquad \text{(A 13)}$$

where $v_1 = U(m+m_0/2)^2$. Upon integration by parts, the first integral is clearly $O(1/t)$. Thus, I_1 reduces to

$$I_1 = -2m_0\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\int_0^\infty \frac{\mathrm{e}^{m^2\psi}}{(m+m_0/2)^2}\,\mathrm{d}m + O\left(\frac{1}{t},\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t\to\infty. \qquad \text{(A 14)}$$

After using (A 11), we can write the second line of (A10) as

$$I_2 = -\mathrm{i}2m_0U^{1/2}\left\{\int_0^{m_2}\frac{W_2(v_2)}{(m_2-m)^{1/2}} + \int_{m_2}^\infty \frac{W_1(-v_2)}{(m-m_2)^{1/2}}\right\}\frac{\mathrm{e}^{m^2\psi^*-\mathrm{i}\Omega_1 t}}{(m+m_1)^{1/2}}\,\mathrm{d}m$$
$$-2m_0\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}⨍_0^\infty \frac{\mathrm{e}^{m^2\psi^*}}{(m+m_1)(m-m_2)}\,\mathrm{d}m + O\left(\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{3/2}}\right), \qquad t\to\infty, \qquad \text{(A 15)}$$

where $m_{1,2}=m_0(\sqrt{2}\pm 1)/2$, $v_2=U(m+m_1)(m_2-m)$, and the path of integration is indented to pass below the pole at $m=m_2$. The integrals inside the brackets have neither poles nor stationary phase points for $m\in(0,\infty)$. Thus, they are dominated by the integration near the end points. Using the method of steepest descent, it can be shown that the integration from the interval near $m=0$ is $O(1/t)$, while it is $O(1/t^{1/2})$ near $m=m_2$. Omitting some details, we express the expansion of I_2 as

$$I_2 = -\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}}\left\{\mathrm{i}2\sqrt{2}\pi\mathrm{e}^{m_2^2\psi^*} + 2m_0⨍_0^\infty \frac{\mathrm{e}^{m^2\psi^*}}{(m+m_1)(m-m_2)}\,\mathrm{d}m\right\} + O\left(\frac{1}{t}\right) \qquad \text{(A 16)}$$

as $t\to\infty$.

For the third line of (A 10), we can rewrite the integral with respect to τ in terms of error functions:

$$I_3 = -\mathrm{e}^{-\mathrm{i}\omega_c t+\mathrm{i}\pi/4}m_0(\pi U)^{1/2}\int_{-\infty}^\infty \frac{\mathrm{e}^{m^2\psi+\mathrm{i}(m-m_0/2)^2Ut}}{|m-m_0/2|}\left[\mathrm{erf}((U_m t)^{1/2}) - \mathrm{erf}((U_m t_0)^{1/2})\right]\mathrm{d}m, \qquad \text{(A 17)}$$

where $U_m=U(m-m_0/2)^2\mathrm{e}^{\mathrm{i}\pi/4}$. Since the error function tends to zero as $m\to m_0/2$, the integral in (A 17) is regular. At large time, the main contribution to this integral comes from the interval near the stationary phase point at $m=m_0/2$. Replacing the first error function in (A 17) by its series expansion and applying the method of steepest descent, we obtain the asymptotic expansion for I_3:

$$I_3 = m_0U^{1/2}\left[\sum_{n=0}^\infty \frac{\Upsilon(n+\frac{1}{2})}{(2n+1)n!} - 2\left(\frac{\pi t_0}{t}\right)^{1/2}\right]\mathrm{e}^{-\mathrm{i}\omega_c t+\kappa\psi-\mathrm{i}\pi/4} + O\left(\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t}\right), \quad t\to\infty, \qquad \text{(A 18)}$$

where Υ represents the gamma function. Here the summation can be shown to be convergent.

After using (A 11) for the integration with respect to τ, the fourth line of (A 10) can be written as

$$I_4 = m_0 \frac{e^{-i\omega_c t}}{t^{1/2}} \int_L \frac{e^{m^2\psi^*}}{(m-m_1)(m+m_2)}\, dm$$
$$+ i m_0 U^{1/2} \left\{ \int_{-\infty}^{-m_2} \frac{W_1(\nu_3)}{(m_1-m)^{1/2}(-m-m_2)^{1/2}} + \int_{-m_2}^{m_1} \frac{W_2(-\nu_3)}{(m_1-m)^{1/2}(m+m_2)^{1/2}} \right.$$
$$\left. + \int_{m_1}^{\infty} \frac{W_1(\nu_3)}{(m-m_1)^{1/2}(m+m_2)^{1/2}} \right\} e^{m^2\psi^* - i\Omega_2 t}\, dm + O\left(\frac{e^{-i\omega_c t}}{t^{3/2}}\right), \quad t \to \infty, \quad \text{(A 19)}$$

where $\nu_3 = U(m-m_1)(m+m_2)$, and the contour L extends from $-\infty$ to $+\infty$ in the complex m-plane and is indented to pass above the pole at $m=-m_2$ and below the pole at $m=m_1$. Clearly, there are no poles for the integrals inside the brackets. The main contribution to these integrals thus comes from the region near the stationary phase point ($m=m_0/2$) and the end points of the integration ($m=m_1, -m_2$). These contributions can be determined using the method of steepest descent. The final result can be expressed as

$$I_4 = \frac{e^{-i\omega_c t}}{t^{1/2}} \left\{ i\sqrt{2}\pi (e^{m_1^2\psi^*} + e^{m_2^2\psi^*}) + m_0 \int_L \frac{e^{m^2\psi^*}}{(m-m_1)(m+m_2)}\, dm \right\}$$
$$+ \frac{e^{i\omega_c t}}{t^{1/2}} (2\pi)^{1/2} W_2(m_0^2 U/2)\, e^{\kappa\psi^* + i\pi/4} + O\left(\frac{e^{-i\omega_c t}}{t^{3/2}}\right), \qquad t \to \infty. \quad \text{(A 20)}$$

In summary, the large-time expansion of Ψ_2 can be expressed as

$$\Psi_2 = \frac{e^{-i\omega_c t}}{t^{1/2}} \left[(C_2 t^{1/2} + \bar{C}_2) e^{\kappa\psi} + G_2(\mathbf{x}; \mathbf{x}') \right] + \hat{C}_2 \frac{e^{i\omega_c t}}{t^{1/2}} e^{\kappa\psi^*} + O\left(\frac{1}{t}, \frac{e^{-i\omega_c t}}{t}\right), \quad \text{(A 21)}$$

where the constants C_2, $\bar{C}_2$, and $\hat{C}_2$ are respectively given by

$$C_2 = e^{-i\pi/4} m_0 U^{1/2} \sum_{n=0}^{\infty} \frac{\Upsilon(n+\frac{1}{2})}{(2n+1)n!}, \quad \text{(A 22)}$$

$$\bar{C}_2 = -2e^{-i\pi/4} m_0 (\pi U t_0)^{1/2}, \qquad \hat{C}_2 = e^{i\pi/4} (2\pi)^{1/2} W_2(m_0^2 U/2), \quad \text{(A 23)}$$

and the time-independent function G_2 is given by

$$G_2(\mathbf{x}; \mathbf{x}') = \ln\left(\frac{r}{r_1}\right) + i\sqrt{2}\pi (e^{m_1^2\psi^*} - e^{m_2^2\psi^*}) - 2m_0 \int_0^{\infty} \frac{e^{m^2\psi}}{(m+m_0/2)^2}\, dm$$
$$- 2m_0 \int_0^{\infty} \frac{e^{m^2\psi^*}}{(m+m_1)(m-m_2)}\, dm + m_0 \int_L \frac{e^{m^2\psi^*}}{(m-m_1)(m+m_2)}\, dm. \quad \text{(A 24)}$$

A.3. *Case III*: $\sigma(\tau) = q(\tau) \neq 0$ *for* $\tau < t_0$; $\sigma(\tau) = 0$ *for* $\tau \geqslant t_0$

If a source is turned off at $\tau = t_0$, the resulting potential Ψ_3 for $t > t_0$ can be expressed as:

$$\Psi_3(\mathbf{x}, t; \mathbf{x}', q) = i2g^{1/2} \int_0^{\infty} e^{m^2\psi + i\Omega_1 t}\, dm \int_{-\infty}^{t_0} q(\tau) e^{-i\Omega_1 \tau}\, d\tau + \text{c.c.}$$
$$- i g^{1/2} \int_{-\infty}^{\infty} e^{m^2\psi + i\Omega_2 t}\, dm \int_{-\infty}^{t_0} q(\tau) e^{-i\Omega_2 \tau}\, d\tau + \text{c.c.} \quad \text{(A 25)}$$

Assuming $q(\tau)$ to be a smooth and continuous function of τ, the integrals with respect to τ in (A 25) are regular. At large time, the integrals with respect to m are then dominated by the integration near the stationary phase point at $m{=}m_0/2$. Upon using the method of stationary phase, we obtain

$$\Psi_3 = C_3 \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^{1/2}} \mathrm{e}^{\kappa\psi} + \text{c.c.} + O\left(\frac{1}{t}\right), \qquad t \to \infty, \tag{A 26}$$

where the constant C_3 is given by

$$C_3 = \mathrm{e}^{-\mathrm{i}\pi/4}(\pi k_0 U)^{1/2} \int_{-\infty}^{t_0} q(\tau)\mathrm{e}^{\mathrm{i}\omega_c \tau}\, \mathrm{d}\tau . \tag{A 27}$$

Appendix B. Large-time expansions of three-dimensional transient single-source potentials

In this appendix, we derive asymptotic expansions of the single-source potential Ψ in three dimensions for three special cases of source strength $\sigma(t)$, for $|\boldsymbol{x}-\boldsymbol{x}'|/Ut = o(1)$ as $t \to \infty$.

B.1. *Case I: $\sigma(\tau) = 0$ for $\tau \leqslant 0$, $\sigma(\tau) = 1$ for $0 < \tau \leqslant t$*

After carrying out the integration with respect to τ in (3.13), we have

$$\begin{aligned}\Psi_1(\boldsymbol{x},t;\boldsymbol{x}') &= \frac{1}{R} - \frac{1}{R_1} \\ &-\frac{4k_0}{\pi}\int_0^{\pi/2} \mathrm{d}\theta ⨍_0^{\infty} \mathrm{e}^{k(z+z')} \cos[k(y-y')\sin\theta] \frac{\cos[k(x-x')\cos\theta]}{k\cos^2\theta - k_0}\, \mathrm{d}k \\ &-\frac{4m_0}{\pi}\int_0^{\pi/2} \mathrm{d}\theta \int_0^{\infty} m\mathrm{e}^{m^2(z+z')} \cos[m^2(y-y')\sin\theta] \frac{\mathrm{e}^{\mathrm{i}(m^2\xi\cos\theta + mtg^{1/2})}}{m\cos\theta + m_0}\, \mathrm{d}m + \text{c.c.} \\ &+\frac{2m_0}{\pi}\int_0^{\pi/2} \mathrm{d}\theta ⨍_{-\infty}^{\infty} m\mathrm{e}^{m^2(z+z')} \cos[m^2(y-y')\sin\theta] \frac{\mathrm{e}^{\mathrm{i}(m^2\xi\cos\theta - mtg^{1/2})}}{m\cos\theta - m_0}\, \mathrm{d}m + \text{c.c.}\end{aligned} \tag{B 1}$$

From (B 1) it is clear that $\Psi_1(\boldsymbol{x},t;\boldsymbol{x}')$ is symmetric about $y-y'{=}0$. Thus, we only need to consider the case $y-y' > 0$.

The first two lines of (B 1) are independent of time and thus a part of the steady solution. For the integration with respect to m in the third line, we see that there exists neither a pole nor a stationary phase point within the range of integration. The leading contribution to this integral comes from the end region $m \in (0,\epsilon)$. Changing the path of integration to the line $m{=}\mathrm{i}\varrho/g^{1/2}$ and making a Taylor series expansion of the integrand about $\varrho{=}0$, the third line of (B1) becomes

$$\begin{aligned} I_2 &= -\frac{8}{\pi}\int_0^{\pi/2} \mathrm{d}\theta \int_0^{\infty}\left[-\frac{\varrho}{m_0^2U^2} + O(\varrho^2)\right]\mathrm{e}^{-\varrho t}\, \mathrm{d}\varrho \\ &= \frac{4}{k_0(Ut)^2} + O\left(\frac{1}{t^3}\right), \qquad t \to \infty . \end{aligned} \tag{B 2}$$

In order to evaluate the integrals in the last line of (B1), we first rewrite it as

$$I_3 = \frac{m_0}{\pi}\int_0^{\pi/2} \sec\theta\, [\Theta(\zeta_1,\theta) + \Theta(\zeta_2,\theta)]\ \mathrm{d}\theta + \text{c.c.}, \tag{B 3}$$

where $\zeta_{1,2} = \xi\cos\theta \pm (y-y')\sin\theta$ and the function Θ is defined by

$$\Theta(\zeta,\theta) = ⨍_{-\infty}^{\infty} \frac{m}{m-\vartheta}\, \mathrm{e}^{m^2(z+z')+\mathrm{i}(m^2\zeta - mtg^{1/2})}\, \mathrm{d}m , \tag{B 4}$$

with $\vartheta = m_0 \sec\theta$. Note that $\zeta_{1,2}$ are non-negative for $|x - x'|/Ut = o(1)$ as $t \to \infty$. In evaluating Θ, therefore, we need to consider the case $\zeta > 0$ only. From (B 4), it is seen that Θ is dominated not only by the pole at $m=\vartheta$ but also by the stationary phase point at $m = tg^{1/2}/2\zeta$. To obtain the asymptotic expansion of Θ at large time, the method of steepest descent is again used. After changing the path of integration to the line of steepest descent given by $m_{1,2} = tg^{1/2}/2\zeta \pm \varrho e^{i\pi/4}$, it follows that

$$\Theta(\zeta,\theta) = i\pi\vartheta e^{\vartheta^2(z+z')+i(\vartheta^2\zeta - \vartheta t g^{1/2})} + \exp\left[i\left(\frac{\pi}{4} - \frac{gt^2}{4\zeta}\right)\right] \int_0^\infty \left[\frac{m_1 e^{m_1^2(z+z')}}{m_1 - \vartheta} + \frac{m_2 e^{m_2^2(z+z')}}{m_2 - \vartheta}\right] e^{-\zeta\varrho^2}\, d\varrho\,, \quad \text{(B 5)}$$

in which the first term results from the residue at the pole $m=\vartheta$. By use of Laplace's method, we can carry out the integral in (B 5) to obtain

$$\Theta(\zeta,\theta) = i\pi\vartheta e^{\vartheta^2(z+z')+i(\vartheta^2\zeta - \vartheta t g^{1/2})} + \frac{t(\pi g)^{1/2}}{tg^{1/2} - 2\vartheta\zeta} \exp\left[i\left(\frac{\pi}{4} - \frac{gt^2}{4\zeta}\right) + \frac{gt^2}{4\zeta^2}(z+z')\right]\left[\frac{1}{\zeta^{1/2}} + O\left(\frac{1}{\zeta^{3/2}}\right)\right], \quad \zeta \to \infty. \quad \text{(B 6)}$$

Substitution of (B 6) into (B 3) gives

$$I_3 = -4\int_0^{\pi/2} \vartheta^2 \sin[m_0\vartheta(x-x')] \cos[\vartheta^2 \sin\theta(y-y')] e^{\vartheta^2(z+z')}\, d\theta + te^{i\pi/4}\left(\frac{g}{\pi}\right)^{1/2} \int_0^{\pi/2} \vartheta \left\{ \frac{\exp\left[-i\frac{gt^2}{4\zeta_1} + \frac{gt^2}{4\zeta_1^2}(z+z')\right]}{\zeta_1^{1/2}(tg^{1/2} - 2\vartheta\zeta_1)} + \frac{\exp\left[-i\frac{gt^2}{4\zeta_2} + \frac{gt^2}{4\zeta_2^2}(z+z')\right]}{\zeta_2^{1/2}(tg^{1/2} - 2\vartheta\zeta_2)} \right\} \times \left[1 + O\left(\frac{1}{\zeta_1}, \frac{1}{\zeta_2}\right)\right] d\theta + \text{c.c.}, \qquad \zeta_1, \zeta_2 \to \infty. \quad \text{(B 7)}$$

After plugging ζ_1 and ζ_2 into (B 7), it becomes clear that the terms inside the brackets have neither poles nor stationary phase points for $\theta \in (0, \pi/2)$. Thus the associated integral is dominated by the integration near $\theta=0$. Taking the end expansion about $\theta=0$ and extending the range of integration to infinity, the resulting integral can be integrated to give

$$I_3 = -4\int_0^{\pi/2} \vartheta^2 \sin[m_0\vartheta(x-x')] \cos[\vartheta^2 \sin\theta(y-y')] e^{\vartheta^2(z+z')}\, d\theta + 2m_0 \frac{(2Ut/\xi)^{1/2}}{tg^{1/2} - 2\xi m_0} \exp\left[-i\frac{gt^2}{4\xi} + \frac{gt^2}{4\xi^2}(z+z')\right]\left[1 + O\left(\frac{1}{\xi}, \frac{1}{t}\right)\right] + \text{c.c.}, \quad t \to \infty. \quad \text{(B 8)}$$

Substituting ξ for (B 8) and writing a Taylor series expansion for $|x - x'|/Ut = o(1)$, I_3 can be expressed as

$$I_3 = -4\int_0^{\pi/2} \vartheta^2 \sin[m_0\vartheta(x-x')] \cos[\vartheta^2 \sin\theta(y-y')] e^{\vartheta^2(z+z')}\, d\theta - \frac{2\sqrt{2}}{U}\frac{e^{-i\omega_c t}}{t} e^{\kappa\varphi} + \text{c.c.} + O\left(\frac{e^{-i\omega_c t}}{t^2}\right), \qquad t \to \infty\,. \quad \text{(B 9)}$$

The asymptotic expansion of $\Psi_1(\boldsymbol{x},t;\boldsymbol{x}')$ is summarized as follows:

$$\Psi_1(\boldsymbol{x},t;\boldsymbol{x}') = \mathscr{G}_1(\boldsymbol{x};\boldsymbol{x}') + \mathscr{C}_1\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t}\mathrm{e}^{\kappa\psi} + \mathrm{c.c.} + O\left(\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^2}\right), \qquad t\to\infty, \tag{B 10}$$

where the constant $\mathscr{C}_1 = -2\sqrt{2}/U$, and the time-independent part $\mathscr{G}_1$ is given by

$$\begin{aligned}\mathscr{G}_1(\boldsymbol{x};\boldsymbol{x}') &= \frac{1}{R} - \frac{1}{R_1}\\ &-\frac{4k_0}{\pi}\int_0^{\pi/2}\mathrm{d}\theta \,⨍_0^{\infty}\mathrm{e}^{k(z+z')}\cos[k(y-y')\sin\theta]\frac{\cos[k(x-x')\cos\theta]}{k\cos^2\theta - k_0}\,\mathrm{d}k\\ &-4k_0\int_0^{\pi/2}\sec^2\theta\sin[k_0\sec\theta(x-x')]\cos[k_0\sec^2\theta\sin\theta(y-y')]\mathrm{e}^{k_0\sec^2\theta(z+z')}\,\mathrm{d}\theta.\end{aligned} \tag{B 11}$$

B.2. *Case II:* $\sigma(\tau)=0$ *for* $\tau\leqslant t_0$, $\sigma(\tau)=\tau^{-1}\mathrm{e}^{-\mathrm{i}\omega_c\tau}$ *for* $t_0<\tau<t$

For a source of varying strength $t^{-1}\mathrm{e}^{-\mathrm{i}\omega_c t}$ for $t>t_0$, the expansion of the resulting velocity potential $\Psi_2(\boldsymbol{x},t;\boldsymbol{x}')$ at large time can be obtained by a procedure similar to that in two dimensions (Case II in Appendix A). We omit the details and give the final result:

$$\begin{aligned}\Psi_2(\boldsymbol{x},t;\boldsymbol{x}') &= \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t}[(\mathscr{C}_2\ln t + \bar{\mathscr{C}}_2)e^{\kappa\psi} + \mathscr{G}_2(\boldsymbol{x};\boldsymbol{x}')] + \hat{\mathscr{C}}_2\frac{\mathrm{e}^{\mathrm{i}\omega_c t}}{t}e^{\kappa\psi^*}\\ &+O\left(\frac{1}{t^2},\frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t^2}\right), \qquad t\to\infty,\end{aligned} \tag{B 12}$$

where the constant $\mathscr{C}_2 = \mathrm{i}2\sqrt{2}\kappa$, and $\bar{\mathscr{C}}_2$ and $\hat{\mathscr{C}}_2$ are respectively given by

$$\bar{\mathscr{C}}_2 = \mathrm{i}2\sqrt{2}\kappa\left[-\ln t_0 + \pi^{-1/2}\sum_{n=1}^{\infty}\frac{\Upsilon(n+1/2)}{n!n}\right], \tag{B 13}$$

and

$$\hat{\mathscr{C}}_2 = -\mathrm{i}2\sqrt{2}\kappa\int_{k_0Ut_0/2}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\lambda}}{\lambda}\mathrm{d}\lambda. \tag{B 14}$$

The time-independent function $\mathscr{G}_2$ is given by

$$\begin{aligned}\mathscr{G}_2(\boldsymbol{x};\boldsymbol{x}') &= \frac{1}{R} - \frac{1}{R_1} + \mathrm{i}2\int_0^{\pi/2}\frac{\cos\theta}{(1+\cos\theta)^{1/2}}[\mathscr{Q}(M_2)-\mathscr{Q}(M_1)]\,\mathrm{d}\theta\\ &+\frac{4m_0}{\pi}\int_0^{\pi/2}\left\{\int_0^{\infty}\frac{\mathscr{Q}^*(m)}{mm_0+m^2\cos\theta+\kappa} + ⨍_0^{\infty}\frac{\mathscr{Q}(m)}{(m+M_1)(m-M_2)}\right.\\ &\left.-2\int_L\frac{\mathscr{Q}(m)}{(m-M_1)(m+M_2)}\right\}\mathrm{d}m\,\mathrm{d}\theta,\end{aligned} \tag{B 15}$$

where the wavenumbers $M_{1,2}$ and the function $\mathscr{Q}$ are given by

$$M_{1,2} = \frac{(1+\cos\theta)^{1/2}\pm 1}{2\cos\theta}m_0, \tag{B 16}$$

$$\mathscr{Q}(m) = m^2\cos[m^2(y-y')\sin\theta]\mathrm{e}^{m^2[(y-y')-\mathrm{i}(x-x')\cos\theta]}. \tag{B 17}$$

B.3. *Case III*: $\sigma(\tau) = q(\tau) \neq 0$ *for* $\tau < t_0$; $\sigma(\tau) = 0$ *for* $\tau \geqslant t_0$

In this case, $\Psi_3(\boldsymbol{x}, t; \boldsymbol{x}', q)$ at large time can be expanded as

$$\Psi_3(\boldsymbol{x}, t; \boldsymbol{x}', q) = \mathscr{C}_3 \frac{\mathrm{e}^{-\mathrm{i}\omega_c t}}{t} \mathrm{e}^{\kappa\psi} + \text{c.c.} + O\left(\frac{1}{t^2}\right), \qquad t \to \infty, \tag{B 18}$$

where the constant $\mathscr{C}_3$ is given by

$$\mathscr{C}_3 = \mathrm{i}2\sqrt{2}\kappa \int_{-\infty}^{t_0} q(\tau)\, \mathrm{e}^{\mathrm{i}\omega_c \tau}\, \mathrm{d}\tau. \tag{B 19}$$

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